International Journal of Advances in Applied Sciences (IJAAS)
Vol. 2, No. 3, September 2013, pp. 125~132
ISSN: 2252-8814

On Stereographic Lognormal Distribution

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Article Info

Article history:

Received Jun 12, 201x
Revised Aug 20, 201x
Accepted Aug 26, 201x

Keyword:

Characteristic function
Circular model
Measurable function
Stereographic projection
Trigonometric moments

ABSTRACT

Circular data arise in various walks of life, where we consider the unit circle to be the sample space. Minh & Farnum (2003) developed that Mobius transformation or Stereographic projection is applied to generate probability distributions on real line [1]. We derive an asymmetric distribution called Stereographic Lognormal distribution to model circular data with an emphasis on the Inverse Stereographic Projection. The graphs of probability density function, cumulative distribution function and characteristic function are drawn. Since Stereographic Lognormal distribution is an asymmetric, a complex-valued characteristic function is derived and population characteristics are also studied. Goodness of fit is verified for the data Set of cross-bed azimuths of palaeocurrents of size 40 which contains angular data.

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1. INTRODUCTION

Dattatreya Rao et al (2007) derived new wrapped models to fit circular data [2]. A popular mathematical transformation viz Mobius transformation / Stereographic Projection is used to propose a new method of generating probability distributions, which maps every point on real line onto the point on unit circle by Minh and Farnum (2003) [1]. Toshihiro Abe et al (2010) applied Inverse Stereographic Projection to develop symmetric circular models [3]. Dattatreya Rao et al (2011) generated Cauchy type models by inducing Stereographic Projection on well known circular distribution viz Cardioid model [4]. Here an attempt is made to generate a new circular model called Stereographic Lognormal distribution on the lines of Minh &Farnum (2003). The characteristic function for the said model is derived to evaluate the trigonometric moments which are required to compute the population characteristics [5]. Goodness of fit is verified for a live data.

2. METHODOLOGY OF INVERSE STEREOGRAPHIC PROJECTION

Inverse Stereographic Projection is defined by a one to one mapping given by $T(\theta) = x = u + v\tan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty)$, $\theta \in [-\pi, \pi)$, $u \in \mathbb{R}$, *and* $v > 0$. Suppose X is randomly chosen on the interval $(-\infty, \infty)$ .Let $F(x)$ and $f(x)$ denote the Cumulative distribution and probability density functions of the random variable *X* respectively. Then $T^{-1}(x)=\theta=2\tan^{-1}\left\{\frac{(x-u)}{v}\right\}$ by Minh and Farnum (2003) is a random point on the Unit circle [1]. Let $G(\theta)$ and $g(\theta)$ denote the Cumulative distribution and probability density functions of this random point θ respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following Theorem 2.1.

Theorem 2.1: For $v>0$,

i) $$G(\theta)=F\left(u+v\tan\left(\frac{\theta}{2}\right)\right)=F\left(x(\theta)\right)$$

ii) $$g(\theta)=v\left(\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{2}\right)f\left(u+v\tan\left(\frac{\theta}{2}\right)\right)$$

3. STEREOGRAPHIC LOGNORMAL DISTRIBUTION

A random variable X on the real line is said to have Lognormal Distribution with location parameter μ and scale parameter $\sigma>0$, if the probability density function and cumulative distribution function of X respectively are given by

1. $$f(x)=\begin{cases}\frac{1}{\sigma\sqrt{2\pi}\,(x-\mu)}\exp\left(-\left(\frac{\ln(x-\mu)}{\sqrt{2}\sigma}\right)^2\right), & x>\mu,\ \sigma>0\\ 0 & \text{if } x\le\mu\end{cases}$$

2. $$F(x)=\frac{1}{2}\left[1+erf\left(\frac{ln(x-\mu)}{\sqrt{2}\sigma}\right)\right],\ \sigma>0$$

Then by applying Inverse Stereographic Projection defined by a one to one mapping $x=v\tan\left(\frac{\theta}{2}\right),\ v>0,\ -\pi\le\theta<\pi$, which leads to Stereographic Lognormal Distribution on unit circle.

Definition:

A random variable Φ on unit circle is said to have Stereographic Lognormal distribution with location parameter μ, scale parameter $\sigma>0$ and concentration parameter $v>0$, denoted by SLN(μ,σ,v), if the probability density and cumulative distribution functions are respectively given by

1. $$g(\theta)=\begin{cases}\frac{\sec^2\left(\frac{\theta-\mu}{2}\right)}{2\sqrt{2\pi}\sigma\left(\tan\left(\frac{\theta-\mu}{2}\right)\right)}\exp\left(-\left(\frac{\ln\left(v\tan\left(\frac{\theta-\mu}{2}\right)\right)}{\sqrt{2}\sigma}\right)^2\right), & v>0,\ \sigma>0,\ \mu\le\theta<\pi+\mu\\ 0 & \text{if } \theta\le\mu\end{cases}$$

2. $$G(\theta)=\frac{1}{2}\left[1+erf\left(\frac{\ln\left(v\tan\left(\frac{\theta-\mu}{2}\right)\right)}{\sqrt{2}\sigma}\right)\right],\ \sigma>0,\ v>0$$

Clearly *g* is pdf of a circular model.

3.1. Graphs of probability density function and cumulative distribution function of Stereographic Lognormal Distribution for various values of σ and v are presented here.

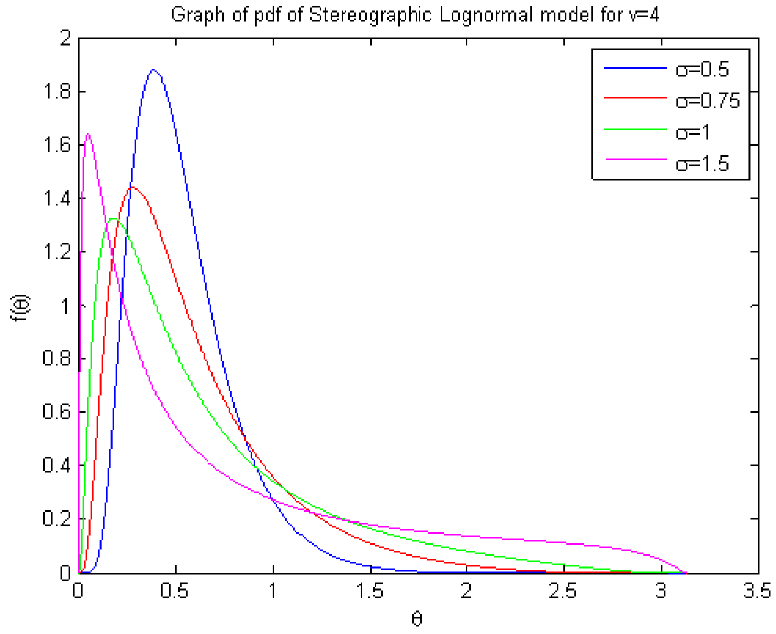

Fig-1

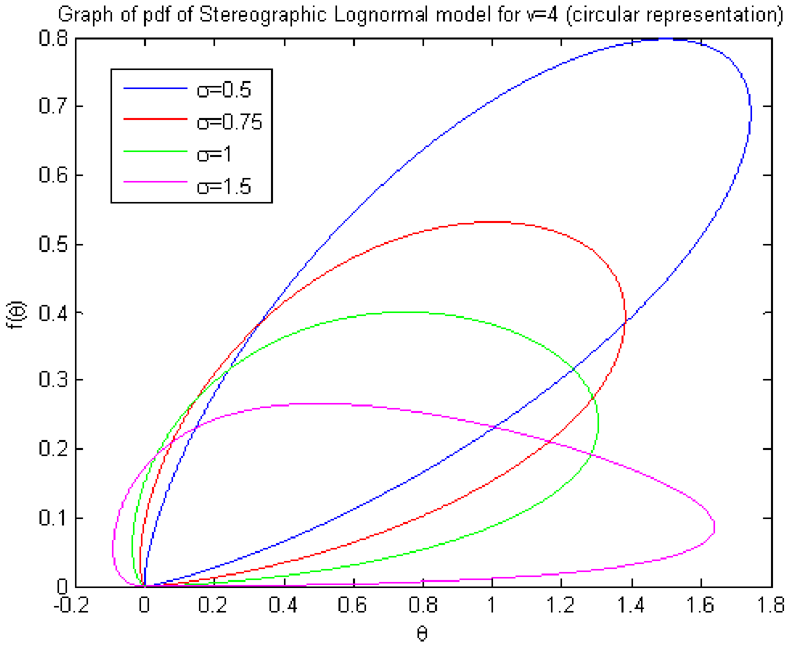

Fig-2

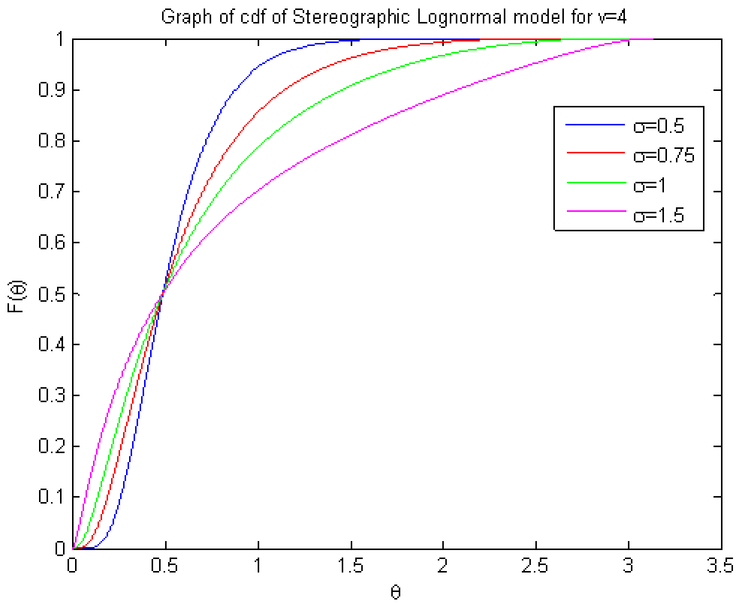

Fig-3

4. THE CHARACTERISTIC FUNCTION OF STEREOGRAPHIC MODELS

The Characteristic function of a Circular model with probability density function $g(\theta)$ is defined as $\varphi_p(\theta) = \int_0^{2\pi} e^{ip\theta} g(\theta) d\theta$, $p \in \mathbb{Z}$. Ramabhadra Sarma et al (2009; 2011) derived the characteristic functions of some new wrapped models [6],[7]. The characteristic function of a Stereographic Circular model can be obtained in terms of respective linear model using the following theorem.

Theorem 4.1 Lukacs (1970) [8]: Let X be a random variable with distribution function $F(x)$ and suppose that $S(x)$ is a finite, single-valued and B-measurable function of *x*. The Characteristic function of $f_Y(t)$ of the random variable $Y = S(x)$ is then given by $f_Y(t) = E\left(e^{itY}\right) = E\left(e^{itS(X)}\right) = \int_{-\infty}^{\infty} e^{itS(X)} dF(x)$.

By applying the above theorem we derive the Characteristic function of a Stereographic Circular model.

Theorem4.2 : If $G(\theta)$ and $g(\theta)$ are the cdf and pdf of the Stereographic Circular model and $F(x)$ and $f(x)$ are cdf and pdf of the respective liner model, then Characteristic function of Stereographic Model is $\varphi_{X_S}(p) = \varphi_{2\tan^{-1}\left(\frac{x}{v}\right)}(p), p \in \mathbb{Z}$

Proof:

$$\varphi_{X_S}(p) = \int_{-\pi}^{\pi} e^{ip\theta} d\left(G(\theta)\right) \quad , p \in \mathbb{Z}$$

$$= \int_{-\pi}^{\pi} e^{ip\theta} d\left(F\left(v \tan\frac{\theta}{2}\right)\right)$$

$$= \int_{-\infty}^{\infty} e^{ip\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)} f(x) dx, \quad \text{taking } x = v\tan\left(\frac{\theta}{2}\right)$$

$$= \varphi_{2\tan^{-1}\left(\frac{x}{v}\right)}(p)$$

4.3 Characteristic function of Stereograph Lognormal Model

Without loss of generality here we assume $\mu = 0$

$$\Phi_{X_S}(p) = \int_0^{\pi} e^{ip\theta} g(\theta)\, d\theta$$

$$= \int_0^{\infty} e^{ip\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)} \frac{1}{\sigma\sqrt{2\pi}\,(x)} e^{-\left(\frac{\log(x)}{\sqrt{2}\sigma}\right)^2} dx$$

$$= \int_0^{\infty} e^{ip\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)} f(x)\, dx$$

$$= \varphi_{2\tan^{-1}\left(\frac{x}{v}\right)}(p)$$

As the integral cannot be obtained analytically, MATLAB Techniques are applied for the evaluation of the values of the characteristic function. The graphs for real and imaginary parts of the characteristic function of the said new model are plotted here using Gauss-Laguerre.

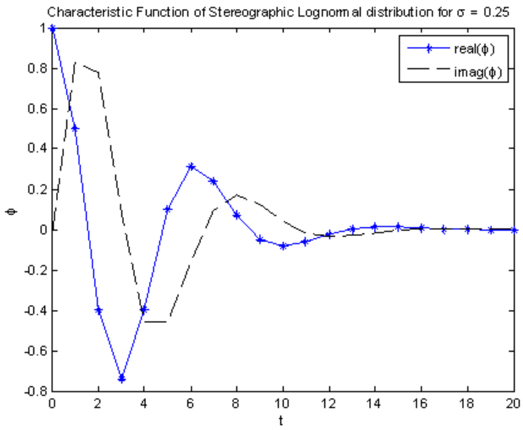

Fig-4

Mardia (2000) gave expressions of mean direction, resultant length, circular variance, circular standard deviation, central trigonometric moments, skewness and kurtosis for circular distributions [5]. These characteristics for the Stereographic Lognormal model are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p and are presented here.

Table1 Characteristics of Stereographic Lognormal distribution for v= 5

Stereographic Lognormal distribution For v = 5.0000		σ = 0.2500	σ = 0.5000	σ = 1.0000	σ = 1.5000	σ = 2
Mean	μ	1.0294	1.0825	0.9798	0.7817	0.6274
Trigonometric	α_1	0.4980	0.4002	0.3403	0.3686	0.3943
Moments	α_2	-0.3998	-0.2449	0.0473	0.1820	0.2559
	β_1	0.8283	0.7533	0.5072	0.3659	0.2859
	β_2	0.7778	0.5525	0.3832	0.3071	0.2548
Resultant length	ρ	0.9665	0.8530	0.6108	0.5193	0.4871
		0.8745	0.6044	0.3861	0.3570	0.3611
Variance	V_o	0.0335	0.1470	0.3892	0.4807	0.5129
Central trigonometric	α_1^*	0.9665	0.8530	0.6108	0.5193	0.4871
Moments	α_2^*	-0.8745	-0.5949	0.3366	0.3085	0.3217
	β_1^*	0.0000	0.0000	0.0000	0.0000	0.0000
	β_2^*	0.0117	0.1063	-0.1891	-0.1797	-0.1640
Skewness	γ_1^o	1.8974	1.8860	-0.7789	-0.5392	-0.4465
Kurtosis	γ_2^o	-1.5530e+003	-52.0033	1.3035	1.0203	1.0088
Circular		1.1815	1.1925	0.9930	1.1447	1.1994
standard deviation	σ_o	1.1985	1.2345	1.3796	1.4353	1.4273

5. GOODNESS OF FIT FOR LIVE DATA

For the purpose of verifying goodness of fit the following **LIVE** data set is considered. Data set is Cross-bed Azimuths of palaeocurrents [9].

Set of cross-bed azimuths of palaeocurrents measured in the Belford Anticline (New South Wales) [10].

284	311	334	320	294	137	123	166	143	127
244	243	152	242	143	186	263	234	209	267
315	329	235	38	241	319	308	127	217	245
169	161	263	209	228	168	98	278	154	279

By applying formulae in Rao Jammalamadaka and Sen Gupta (2001) the mean direction (μ) and scale parameter (σ) are estimated from this data as 3.9804 and 0.4049 respectively [11]. **In this case the no. of observations satisfying the condition $\theta > \mu$ are 21, hence in further computations n is taken as 21 though the original sample size is 40.** Substituting data set in cdf of Stereographic Lognormal model corresponding uniform $(\mu, \pi+\mu)$ variates denoted by $\theta_1, \theta_2, ..., \theta_n$ are obtained. Using these θ_i 's , the tests statistics of Rayleigh Test, Kuiper's Test, Watson's U^2- Test, Hodges –Ajne Test, Range Test, Rao's Equal Spacing Test and Ajne Test are computed and are tabulated in Table 2.

Table 2 : Statistic Values of Various Test Procedures

Tests	Test Statistic for Sample Size (n) = 21
Rayleigh Test	**7.8804**
Kuiper's Test	2.9107
Watson's U^2 - Test	**0.0521**
Hodges –Ajne Test	16
Range Test	**4.7553**
Rao's Equal Spacing Test	**3.4945**
Ajne Test	4.3992

The cut of points for the above sample sizes are taken from Devaraaj (2012) [12].

Table 3: Cut of points for Stereographic Lognormal of sample size n = 21

LOS / Tests	1%	5%	10%
Rayleigh Test	**0.0090 - 9.9306**	0.0449 - 7.6058	0.0979 - 5.9160
Kuiper's Test	0.6990 - 2.1200	0.7893 - 1.8521	0.8288 - 1.7453
Watson's U^2 - Test	**0.0150 - 0.3048**	**0.0203 - 0.2132**	**0.0242 - 0.1814**
Hodges –Ajne Test	2.0000-8.0000	3.0000-7.0000	3.0000-7.0000
Range Test	**3.9980 - 5.6940**	**4.4727 - 5.6122**	**4.6404 - 5.5687**
Rao's Equal Spacing Test	**1.4080 - 3.5034**	1.6172 - 3.0112	1.7343 - 2.8750
Ajne Test	0.0280 - 1.0639	0.0398 - 0.8167	0.0500 - 0.6368

Remark: Stereographic Lognormal model appears to be good fit at at 1%, 5% and 10% based on Watson's U^2- Test and Range test. It is also good fit at 1% with respect to Rayleigh test and Rao's Equal Spacing test.

ACKNOWLEDGEMENTS

Authors would like to thank UGC, New Delhi, India for offering financial assistance to carry out the project under the head of Major Research Project no. F 41-785/2012 (SR) dt. 17-07-2012.

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