# New approach of the neighborhood structure of fuzzy points

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**ABSTRACT**

This paper provides a comparative analysis of the fuzzy Q-neighborhood and the fuzzy neighborhood system of a fuzzy point. Specifically, we investigate the relationship between the elements of these systems when both are defined at the same fuzzy point. We address questions such as: how are these elements interconnected, and which system contains the other? Furthermore, we give the dual of the fuzzy Q-neighborhood system, which is named the fuzzy DQ-neighborhood system, and prove that these two systems are not equivalent. Finally, we examine the properties of these systems to determine whether they satisfy the conditions of fuzzy topology, Supra topology, or filter theory.



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## 1. INTRODUCTION

The principles of fuzzy systems were introduced by Zadeh [1], subsequently influencing various mathematical disciplines such as algebra [2], topology [3], and geometry [4]. Among these, topology has seen the most significant development, which was later called fuzzy topology. It is defined through two primary approaches: those of Chang [3] and Lowen [5]. In this work, we adopt Chang's framework [3], which defines a fuzzy topology on the set X as a family $T \subseteq I^X$, where $I = [0,1]$, satisfying the well-known axioms. This approach has been extensively studied by researchers such as Šostak [6], Ramadan [7], and Vadivel [8].

Research in this field has primarily aimed to explore new structures that expand existing mathematical horizons by generalizing foundational concepts or uncovering novel relationships. Given that fuzzy topology, as a discipline, is relatively young, originating in 1965, its development has proceeded from foundational topics like topology, through fuzzy systems. Establishing and defining these core concepts inevitably generates new questions and problems, such as understanding the relationship between different concepts, determining whether one concept leads to another, and identifying the conditions required for such transitions. Additionally, weak these conditions can yield new concepts that demand further exploration.

The concept of fuzzy neighborhood holds a significant place in mathematics, and extends its effect into fields as computer science and engineering, particularly in the context of data modeling [9]–[11]. Therefore, it is essential to explore the topic from multiple angles. One such aspect involves examining the various types of fuzzy neighborhoods, among which the fuzzy Q-neighborhood [12], [13] is of particular interest. During our study, we note that the dual of the fuzzy Q-neighborhood has received little to no attention in prior research. Questions naturally arise: what form does this dual take?, is it possible to construct a fuzzy topological category based on it?, and these are some of the central inquiries we aim to explore and address throughout this study.

Our study presents an approach to the fuzzy Q-neighborhood of a fuzzy point [12], specifically investigating its relationship with the fuzzy neighborhood of a fuzzy point [14]–[16]. Previous studies have largely overlooked this specific relationship in their analyses, although prior research has examined fuzzy neighborhood systems through consistent functions and quantale-based structures as [17]–[19]. In contrast, this paper brings it to the forefront, offering a clear explanation supported by illustrative examples. At the same time, we introduced the dual of the fuzzy Q-neighborhood concept and clarified how it relates to previously established fuzzy neighborhood structures. We also examine whether relaxing certain conditions leads to new insights and explore the connections between these systems. Through discussion, examples, and theoretical analysis. Exploring the category-theoretic approach to fuzzy dual Q-neighborhood can lead to the development of a new categorical structure that differs fundamentally from the fuzzy filter framework typically induced by fuzzy neighborhoods and Q-neighborhood.

## 2. PRELIMINARIES

We will recall some concepts that we need in this paper. It is important to understand the paper procedures. From now on X always denotes a nonempty set called the universal set, its elements have membership 1, and A is a fuzzy subset of X, which we will call later fuzzy set. On the contrary, the empty set $\emptyset$ whose elements have membership 0. Also, don't forget, I denotes [0,1] where $0 \leq \mu_A(x) \leq 1$, for each A is a fuzzy set in X.

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Definition 2.1: a fuzzy set A in universal set X is a collection of ordered pairs as $A = \{(x, \mu_A(x)): x \in X\}, \mu_A(x) \in I$, [1], [2]. Definition 2.2: a fuzzy point $x_t, (0 < t \leq 1)$, in X is a fuzzy set which takes the value 0 for all $y \in X$ except for one element, say $x \in X$, [1], [3]. Definition 2.3: let $x_t$ be fuzzy points and A, B are fuzzy sets in X [1], [3] then we say: i) $x_t \in A$ iff $t \leq \mu_A(x)$. Other than that $x_t \notin A$; ii) $B \subseteq A$ iff $\mu_B(x) \leq \mu_A(x), \forall x \in X$. Other than $B \nsubseteq A$; iii) $A = B$ iff $\mu_A(x) = \mu_B(x), \forall x \in X$. Other than $A \neq B$; iv) the union of fuzzy sets $A_i \subseteq X$ has a membership function which defined for each $x \in X$ by: $\mu_{\cup_{i\in I} A_i}(x) = \max\{\mu_{A_i}(x), i \in I\}$; v) the intersection of fuzzy sets $A_i \subseteq X, i = 1,2,,\dots,n$ has membership function which defined for each $x \in X$ by by $\mu_{\cap_{i=1}^{n} A_i}(x) = \min\{\mu_{A_i}(x), i = 1,2,,\dots,n\}$; and vi) the complement of a fuzzy set $A \subseteq X$ has a membership function which defined for each $x \in X$ by: $\mu_{A^c}(x) = 1 - \mu_A(x)$.

Definition 2.4: the support of subset A $(\sup A)$ is the crisp set of all elements in X that have a non-zero membership, i.e., $\sup A = \{x \in X: \mu_A(x) > 0\}$, [1], [4]. Theorem 2.1: let A and B be two fuzzy sets in the universal set X [4], [5], then: i) $A \subseteq B$ iff for each $x_t \in A$ then $x_t \in B$; ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$; iii) $x_t \in A \cup B$ iff $x_t \in A$ or $x_t \in B$; and iv) $x_t \in A \cap B$ iff $x_t \in A$ and $x_t \in B$. However, there doesn't need to exist a relation between the formulas $x_t \in A$ and $x_t \in A^c$ and we have $x_t \in A^c$iff $t \leq 1 - \mu_A(x)$, $x_t \notin A^c$ iff $t > 1 - \mu_A(x)$.

Definition 2.5: let X be a universal set [6], [7], then: i) a fuzzy point $x_t$ in X is said to be quasi-coincident with the fuzzy set A in X, denoted by $x_t qA$, if and only if $t + \mu_A(x) > 1$, and not quasi-coincident with the fuzzy set A, denoted by $x_t \leftharpoonup qA$, if and only if $t + \mu_A(x) \leq 1$; and ii) A fuzzy set A in X is said to be quasi-coincident with the fuzzy set B in X, denoted by AqB, iff there exists x in X such that $\mu_A(x) + \mu_B(x) > 1$ and not quasi-coincident with the fuzzy set B in X, denoted by $A \leftharpoonup qB$, iff for each x in X, then $\mu_A(x) + \mu_B(x) \leq 1$.

Theorem 2.2: consider X as a universal set, and let A and B represent fuzzy subsets within X [8], [20], then: i) $A \leftharpoonup qB$ iff $A \subseteq B^c$; ii) $A \subseteq B$ iff $x_t qB, \forall x_t qA$; iii) $A \leftharpoonup qA^c$ For every fuzzy set A defined within X; iv) $A \cap B = \emptyset$ then $A \leftharpoonup qB$; and v) $x_t \leftharpoonup qA$ iff $x_t \in A^c$.

Remark 2.1: note that if AqB then BqA and vice versa, and so $A \leftharpoonup qB$. Now, we will introduce the concepts of fuzzy topological spaces, which we will need later. Definition 2.6: a fuzzy topology T, is defined as a collection of fuzzy sets on a universal set X that satisfies the following conditions [3]: i) $\emptyset, X \in T$; ii) if $A, B \in T$ then $A \cap B \in T$; and iii) if $A_i \in T$ for every $i \in I$, then $\cup_{i\in I} A_i \in T$. T is called a fuzzy topology for X, and the pair $(X, T)$ It is called a fuzzy topological space. Every member of T is called a fuzzy open set. A fuzzy set is closed iff its complement is open.

Definition 2.7: in a fuzzy topological space $(X, T)$, a fuzzy set A is considered a fuzzy neighborhood of a fuzzy point $x_t$ if there exists a fuzzy open set U such that $x_t \in U \subseteq A$. The collection of all fuzzy neighborhoods of $x_t$ is referred to as the fuzzy neighborhood system at $x_t$ and is denoted by $N_{x_t}$, [3]. Example 2.1: let $X = \{x, y\}$ and $T = \{\emptyset, X, \{x_{0.5}\}, \{y_{0.6}\}, \{x_{0.5}, y_{0.6}\}\}$, T is a fuzzy topology on X. Let $A = \{x_{0.6}, y_{0.5}\}$ then A is a fuzzy neighborhood on $x_t, \forall t \leq 0.5$, but A is an impossible fuzzy neighborhood on $y_t, 0 < t \leq 1$. Theorem 2.3: a fuzzy set A is considered fuzzy open iff for every fuzzy point $x_t$ contained in A, A is a fuzzy neighborhood of $x_t$, [3]. Example 2.2: it's clear that A in example (2.1) is not fuzzy open. Definition 2.8: let A be a fuzzy set in a fuzzy topological space $(X, T)$. The interior $A^{\circ}$ and closure $\overline{A}$ are defined as (1) and (2) [21], [22].

$$A^{\circ} = \bigcup\{G: G \text{ is fuzzy open and } G \subseteq A\} \quad (1)$$

$$\bar{A} = \bigcap\{H: H \text{ is fuzzy closed and } A \subseteq H\} \quad (2)$$

Example 2.3: note in example (2.1), $A^{\circ} = \{x_{0.5}\}$ and $\overline{A} = X$. Definition 2.9: a fuzzy filter $\mathcal{F}$ on a universal set X is a collection of fuzzy sets in X that satisfy these conditions [23], [24]: i) If $F_1$ and $F_2$ are elements of $\mathcal{F}$, then $F_1 \cap F_2$ must also belong to $\mathcal{F}$; ii) If F is an element of $\mathcal{F}$ and $F \subseteq G$, then G must also belong to $\mathcal{F}$; and iii) $\emptyset \notin \mathcal{F}$.

Remark 2.2: from 3 in the definition above, then the filter is impossible in topology and vice versa. Example 2.4: the fuzzy neighborhood system $N_{x_t}$ of any fuzzy point $x_t$ in a universal set X is fuzzy filter. Proof: i) let $G_{x_t} \& V_{x_t} \in N_{x_t} \Longrightarrow \exists B \& C \text{ in } T \text{ s.t } x_t \in B \subseteq G_{x_t} \& x_t \in C \subseteq V_{x_t} \Longrightarrow x_t \in B \cap C \subseteq G_{x_t} \cap V_{x_t} \Longrightarrow G_{x_t} \cap V_{x_t} \in N_{x_t}$; ii) If $G_{x_t} \in N_{x_t} \Longrightarrow \exists B$ in T s.t $x_t \in B \subseteq G_{x_t}$ and if $G_{x_t} \subseteq M \Longrightarrow M \in N_{x_t}$ since $x_t \in B \subseteq M$; iii) $\emptyset \notin N_{x_t}$ since $\emptyset$ it has no element with $t > 0$; iv) Definition 2.10: a fuzzy Supra topology $\delta \subseteq I^X$ is defined as follows [25]:
- $\emptyset, X \in \delta$.
- If $B_j \in \delta, \forall j \in J$, then $\bigcup_{j \in J} B_j \in \delta$.

Remark 2.3: from the definition above, every fuzzy topology is a fuzzy Supra topology, but the converse is not true in general, as follows: example 2.5: in example 2.1, let $\delta = \{\emptyset, X, \{x_{0.5}, y_{0.6}\}, \{x_{0.6}, y_{0.5}\}, \{x_{0.6}, y_{0.6}\}\}$ is a fuzzy Supra topology on X, but it is not a fuzzy topology. Remark 2.4: as pointed out in definition 2.9 (3), that fuzzy Supra topology is impossible is fuzzy topology, and so the converse is true.

## 3. Q-NEIGHBORHOODS OF A FUZZY POINT

In this section, we first highlight some of the claims made about the fuzzy Q-neighborhood of a fuzzy point. We then explore its relationship with the fuzzy neighborhood, which is a preliminary step towards constructing a new neighborhood in the next chapter. This will encourage us to infer new relationships and theorems, which we will support with examples.

Definition 3.1 [12]: a fuzzy set A in $(X, T)$ is referred to as a fuzzy quasi-neighborhood (or fuzzy Q-neighborhood) of $x_t$ iff there exists a $B \in T$ such that $x_t q B \subseteq A$. The collection of all fuzzy Q-neighborhoods of $x_t$ is called the system of fuzzy Q-neighborhoods of $x_t$ and denoted by $QN_{x_t}$. Note 3.1: i) a fuzzy Q-neighborhoods of a fuzzy point $x_t$ general may be containing the point itself $x_t$ or no; ii) a fuzzy set A that quasi-coincides with a fuzzy point $x_t$ generally not necessary, it is fuzzy Q-neighborhood of $x_t$; iii) in the following, we will give examples for tow notes above. Examples 3.1: in example 2.1: i) we note that the fuzzy set A is a fuzzy Q-neighborhood of the fuzzy point $x_{0.7}$ since $x_{0.7} q\{x_{0.5}\} \subseteq A$ but $x_{0.7} \notin A$, while A is the fuzzy Q-neighborhood of $x_{0.6}$ since $x_{0.6} q\{x_{0.5}\} \subseteq A$ and $x_{0.6} \in A$; and ii) We note A is quasi-coincident with $y_{0.6}$ but A is not a fuzzy Q-neighborhood of a fuzzy point $y_{0.6}$ since $\nexists B \in T \text{ s.t} y_{0.6} q B \subseteq A$ .

Note 3.2: we also note that the concepts of fuzzy neighborhood and fuzzy Q-neighborhood are separate as follows: example 3.2: in example 2.1, A is the fuzzy Q-neighborhood of the fuzzy point $x_{0.6}$ since $x_{0.6} q\{x_{0.5}\} \subseteq A$ but is not a fuzzy neighborhood since $\nexists B \in T \text{ s.t} x_{0.6} \in B \subseteq A$ and so A is a fuzzy neighborhood of the fuzzy point $x_{0.4}$ since $x_{0.4} \subseteq \{x_{0.5}\} \subseteq A$, but it is not fuzzy Q-neighborhood since $\nexists B \in T \text{ s.t } x_{0.4} q B \subseteq A$. Remark 3.1: we observe that if a fuzzy set is not quasi-coincident with a fuzzy point, then it is impossible for it to be the fuzzy Q-neighborhood of that point. Proposition 3.1: For any $x_t \in X$, where $0 < t \leq 1$, then: i) $QN_{x_t}$ is not fuzzy topology or fuzzy Supra topology; and ii) $QN_{x_t}$ is a fuzzy filter on X.

Proof: from remark 3.1 $\emptyset \notin QN_{x_t}$. Let $G, V \in QN_{x_t}$, then $\exists B, C$ in T s.t. $x_t q B \subseteq G$ and $x_t q C \subseteq V \Rightarrow x_t q B \cap C \subseteq G \cap V \Rightarrow G \cap V \in QN_{x_t}$. Let $G \in QN_{x_t} \Rightarrow \exists B \in T \text{ s.t. } x_t q B \subseteq G$ and if $G \subseteq V \Rightarrow x_t q B \subseteq V \Rightarrow V \in QN_{x_t}$. From proof 1, $\emptyset \notin QN_{x_t}$.

From note 3.2 and example 3.2, we know the concepts of fuzzy neighborhood system and fuzzy Q-neighborhoods are separate, but a question might arise in our minds: what is the nature of the relation between the elements in the fuzzy neighborhood and fuzzy Q-neighborhood system at the same fuzzy point?. Theorem 3.1: let $x_t \in X$ be a fuzzy point and let $N_{x_t}$ and $QN_{x_t}$ are fuzzy neighborhoods and fuzzy Q-neighborhood system at $x_t$, respectively, then each element $A \in N_{x_t}$ is quasi-coincident with each $B \in QN_{x_t}$ and so converse. Proof: let $A \in N_{x_t} \Rightarrow \exists U$ is fuzzy open s.t. $x_t \in U \subseteq A$. Let $B \in QN_{x_t} \Rightarrow \exists V$ is fuzzy open s.t. $x_t qV \subseteq B$ and same time.

$$x_t qB \Rightarrow t + \mu_B(x) > 1 \tag{3}$$

But, $x_t \in A \Rightarrow t \le \mu_A(x)$ then from (1) $\mu_A(x) + \mu_B(x) > 1 \Rightarrow AqB$ and so by theorem 2.1 BqA. Theorem 3.2: for any $x_t \in X$ then, $N_{x_t} \subseteq QN_{x_t}$ or $QN_{x_t} \subseteq N_{x_t}$. Proof: let $x_t \in X$. Then $t \le 0.5$ or $t > 0.5$. If $t \le 0.5$, let $B \in QN_{x_t}$ then $\exists V$ is open s.t. $x_t qV \subseteq B \Rightarrow \mu_V(x) > 0.5 \Rightarrow x_t \in V \subseteq B \Rightarrow B \in N_{x_t} \Rightarrow QN_{x_t} \subseteq N_{x_t}$. Now, if $t > 0.5$, let $A \in N_{x_t} \Rightarrow \exists U$ is a fuzzy open set s.t. $x_t \in U \subseteq A$ and since $t > 0.5 \Rightarrow \mu_U(x) + t > 1 \Rightarrow x_t qU \subseteq A \Rightarrow A \in QN_{x_t} \Rightarrow N_{x_t} \subseteq QN_{x_t}$. Theorem 3.3: for any $x_t \in X$. Any system $\Gamma$ which is a fuzzy neighborhood and a Q-neighborhood system on $x_t$ then every element has membership at x larger than 0.5. Proof: let $x_t \in X$, $t \le 0.5$ or $t > 0.5$. If $t \le 0.5$ and $\varsigma \in \Gamma \Rightarrow \varsigma \in QN_{x_t}$ and by theorem 3.2 $\Rightarrow \mu_\varsigma(x) > 0.5$, and if $t > 0.5 \Rightarrow \varsigma \in N_{x_t} \Rightarrow \mu_\varsigma(x) > 0.5$.

## 4. DQ-NEIGHBORHOOD OF A FUZZY POINT

In this section, we will present and prove that the dual isn't equivalent to fuzzy Q-neighborhood. Definition 4.1: a fuzzy set A in fuzzy topological space $(X, T)$ is called a fuzzy dual quasi-neighborhood (in brief, fuzzy DQ-neighborhood) of a fuzzy point $x_t$ iff $x_t qA$ and there exists $V \in T$ s.t. $x_t qV \subseteq A^c$. The family of all the fuzzy DQ-neighborhoods at a fuzzy point $x_t$ is called the fuzzy DQ-neighborhood system of $x_t$, which is denoted by $ND_{x_t}$. Remark 4.1: from definition 2.1, we note the following: i) if A is a fuzzy DQ-neighborhood of $x_t$ then $x_t qA^c$; ii) A is a fuzzy DQ-neighborhood of $x_t$ iff $x_t qA$ and $A^c$ is the fuzzy Q-neighborhood of $x_t$; and iii) If A is a fuzzy Q-neighborhood and DQ-neighborhood of $x_t$ then so $A^c$. Proof: the proof, being straightforward, is omitted. The following example explains that the two concepts, fuzzy DQ-neighborhood and-neighborhoods are not equivalent.

Example 4.1: let $X = \{a, b\}$ and $T = \{\emptyset, X, \{a_{0.4}, b_{0.4}\}\}$. Let $A_1 = \{a_{0.3}, b_{0.3}\}, A_2 = \{a_{0.7}, b_{0.7}\}$ and $A_3 = \{a_{0.5}, b_{0.5}\}$ then $A_1$ is the fuzzy DQ-neighborhood of $a_{0.9}$ but it is not fuzzy Q-neighborhood of $a_{0.9}$. $A_2$ is the fuzzy Q-neighborhood of $a_{0.9}$ but it is not fuzzy DQ-neighborhood of $a_{0.9}$. $A_3$ is a fuzzy DQ-neighborhood and Q-neighborhood of $a_{0.9}$. Note 4.1: a fuzzy DQ-neighborhood set, along with the complement of a fuzzy point, does not include the fuzzy point itself. The neighborhood structure of a point that does not contain the point itself has been explored in [12]. Theorem 4.1: a fuzzy DQ-neighborhood system $ND_{x_t}$ of a fuzzy point $x_t$ in a fuzzy topological space $(X, T)$ satisfies the following: i) $\emptyset, X \notin ND_{x_t}$; ii) If $A_j \in ND_{x_t}, j \in J$ then $\bigcap_{j\in J} A_j \in ND_{x_t}$ for any index J; and iii) If $A_l \in ND_{x_t}, l \in L$ then $\bigcup_{l\in L} A_l \in ND_{x_t}$ for any finite index L. Proof: let $(X, T)$ be a fuzzy topological space and $x_t \in X$ fixed fuzzy point. $\emptyset \leftharpoondown qx_t, \forall x_t \in X \Rightarrow \emptyset \notin ND_{x_t}$. Now if $X \in ND_{x_t} \Rightarrow \emptyset q x_t$ and this contradiction, therefore, $X \notin ND_{x_t}$. For any index J, let $A_j \in ND_{x_t}, j \in J$ then $\exists U_j \in T$ such that $x_t qU_j \subseteq {A_j}^c \Rightarrow x_t q \bigcup_{j\in J} U_j \subseteq \bigcup_{j\in J} {A_j}^c \Rightarrow \bigcup_{j\in J} {A_j}^c$ is the Q-neighborhood of $x_t$, but $A_j q x_t, \forall j \in J \Rightarrow \bigcap_{j\in J} A_j q x_t$ and by Remark 4.1,2$\Rightarrow \bigcap_{j\in J} A_j \in ND_{x_t}$. For any finite index L, let $A_l \in ND_{x_t}, l \in L$ then $\exists V_l \in T$ such that $x_t qV_l \subseteq A_l \Rightarrow x_t q \bigcap_{l\in L} V_l \subseteq \bigcap_{l\in L} {A_l}^c \Rightarrow \bigcap_{l\in L} {A_l}^c$ is the Q-neighborhood of $x_t$, but $A_l q x_t, \forall l \in L \Rightarrow \bigcup_{l\in L} A_l q x_t$ and by Remark 4.1,2$\Rightarrow \bigcup_{l\in L} A_l \in ND_{x_t}$.

In fact, from theorem 4.1, $ND_{x_t}$ it isn't fuzzy topology or fuzzy Supra topology. Furthermore, it is not a fuzzy filter, and this is the opposite of $N_{x_t}$ which proves it earlier. The following example explains that: example 4.2: in example 2.1, $A_1$ is the fuzzy DQ-neighborhood of $a_{0.9}$, i.e., $A_1 \in ND_{a_{0.9}}$ and $A_1 \subseteq A_2$ but $A_2 \notin ND_{a_{0.9}}$. Proposition 4.1: in a fuzzy DQ-neighborhood system $ND_{x_t}$ of a fuzzy point $x_t$ in a fuzzy topological space $(X, T)$, let $A \in ND_{x_t}$ and $B \subseteq A$, if $x_t qB$ then $B \in ND_{x_t}$. Proof: Since $B \subseteq A$, then $A^c \subseteq B^c$. $x_t qU \subseteq A^c \subseteq B^c$, for some $U \in T \Rightarrow B \in ND_{x_t}$.

## 5. RESULTS AND DISCUSSION

The result of this study shows that the fuzzy Q-neighborhood system is also a fuzzy filter. It is worth observing here that the concepts of fuzzy Q-neighborhood and fuzzy neighborhood systems, although unlike each other, are structurally related; the former is limited in the latter and vice versa. This interrelationship has been relatively neglected in literature [10]–[13], although its original interpretation occurs in the modelling of the behavior of fuzzy topological objects.

The main contribution of this paper is the definition and study of the duality of the fuzzy Q-neighborhood system in relation to a fixed fuzzy point. We have shown that this two-class system is generally not a point-non-inclusion or fuzzifying system. This deviation from traditional structures includes the methodology of new theoretical understandings, especially when it comes to understanding neighborhood systems that do not depend on a specific reference point. By distinguishing between original and dual systems, we can classify fuzzy spaces in innovative ways, highlighting the variation of behaviors that can arise in fuzzy frameworks. Furthermore, a detailed study of the connections between fuzzy neighborhood systems, fuzzy Q-neighborhood systems, and their doubles revealed a more complex structure. This research highlights the promise of adopting categorical methods, such as DQ neighborhood systems, to create special mathematical spaces. These systems could provide a richer understanding of concepts such as proximity, continuity, and separation in fuzzy environments.

The study raises some interesting questions about neighborhoods that do not fit the usual fuzzy point model. These kinds of systems are counterproductive to traditional ideas and highlight the need for more research into practical, real-world models of confusion. In the future, future research could explore these structures within categorical, spatial, or algebraic frameworks, paving the way for broader and more applicable theories in fuzzy topology and related areas.

## 6. CONCLUSION

The fuzzy Q-neighborhood system is a fuzzy filter. For any given fuzzy point, it contains either the fuzzy neighborhood system or contains within it an important relationship that has remained largely unknown in previous studies. In this paper, we address this gap by providing a clear analysis supported by illustrated examples. Furthermore, we introduced the dual of the fuzzy Q-neighborhood system, demonstrating that it neither forms a fuzzy filter nor coincides with the original fuzzy Q-neighborhood system. As families of fuzzy sets, these systems provide a means of characterizing specific classes of fuzzy spaces, thus facilitating broader generalizations and deeper structural insights.

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C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject administration
Fu : **Fu**nding acquisition

## CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

## DATA AVAILABILITY

Data availability is not applicable to this paper as no new data were created or analyzed in this study.

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