International Journal of Advances in Applied Sciences (IJAAS)
Vol. 15, No. 1, March 2026, pp. 1∼18
ISSN: 2252-8814, DOI: 10.11591/ijaas.v15.i1.pp1-18

Extension of Hermite-Hadamard type inequalities to Katugampola fractional integrals

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Article Info

Article history:

Received Jun 8, 2025
Revised Sep 28, 2025
Accepted Nov 7, 2025

Keywords:

Exponentially (s,m)-convex function
Fractional calculus
Hölder inequality
Hermite-Hadamard inequality
Katugampola fractional integral
Power-mean inequality

ABSTRACT

In this study, we introduce several new Hermite-Hadamard type general integral inequalities for exponentially (s,m)-convex functions via Katugampola fractional integral. The Katugampola fractional integral is a broader form of the Riemann–Liouville and Hadamard fractional integrals. We utilized the power-mean integral inequality, the Hölder inequality and a few additional generalizations to derive these inequalities. Numerous limiting results are derived from the main results presented in the remarks. Furthermore, we provide an example illustrating our theoretical findings, supported by a graphical representation.

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1. INTRODUCTION

Over the past century, the concept of “convexity” has garnered considerable interest among mathematicians. This term has played a significant role and has received remarkable attention from numerous researchers in the advancement of various fields within pure and applied sciences. In financial mathematics, mathematical statistics, and functional analysis, the theory of convexity holds significant importance. Convex function optimization has numerous real-world uses, such as controller design, circuit design, and modeling. Due to its broad relevance and many practical applications, convexity has developed into a highly influential and intellectually engaging area for scientists and mathematicians. We encourage interested readers to see the references [1]-[6] for some discussion about convexity and its properties.

Inequalities and the property of convexity play a crucial role in contemporary mathematical research. These two concepts are interrelated. Inequalities are essential in diverse areas like mechanics, functional analysis, probability theory, numerical methods, and statistical problems. From this perspective, the field of inequalities stands as an independent discipline in mathematical analysis. For further details, see [7]-[10].

Research on mathematical inequalities associated with fractional integral operators, including the Riemann–Liouville type, plays a key role in the field of fractional calculus. These inequalities play an important role in revealing the behavior of fractional integrals and supporting their application in diverse areas, including physics, engineering, and mathematics. The Riemann-Liouville fractional integral operator, which generalizes the traditional notion of integration to non-integer orders, is fundamental in formulating these inequalities. Developing inequalities for Riemann-Liouville $(R-L)$ fractional integrals allows researchers to establish strict bounds and conditions that are essential for solving fractional differential equations, helping to ensure the stability and robustness of solutions in systems where conventional calculus proves inadequate, see [11]-[14].

By integrating the concepts of fractional calculus with generalized convexity, researchers can formulate a wider range of mathematical inequalities. These inequalities enhance the theoretical comprehension of fractional integrals and also hold practical significance for analyzing and optimizing complex systems in science and engineering. The interaction of generalized convexity with fractional integral operators creates new research opportunities and makes it possible to create stronger mathematical frameworks for fractional dynamics modeling, analysis, and problem solving.

Within this analytical framework, the fractional Hermite-Hadamard inequalities evolve as a key extension of their classical counterparts into the fractional calculus domain. The classical Hermite-Hadamard inequality gives estimates for the integral of a convex function, and its fractional analogue generalizes this idea to the setting of fractional integrals, particularly those defined by operators such as the Riemann–Liouville integral. Fractional Hermite-Hadamard inequalities apply the principles of convexity and fractional calculus to derive bounds for the fractional integrals of convex functions. This inequality is used in many different areas of economics; for example, the existence and uniqueness of certain economic models (such as general equilibrium models or company behavior models) are demonstrated using this inequality. It can also play an important role in various areas of mathematics, such as number theory, complex analysis, and numerical analysis. This inequality can also apply to information theory, engineering, physical science, biology, and chemistry. According to this inequality, let $\Phi_1 : \xi \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval ξ ($\xi \subset \mathbb{R}$) and $w_1, v_1 \in \xi$ with $w_1 < v_1$. The following double inequality

$$\Phi_1\left(\frac{w_1+v_1}{2}\right) \leqslant \frac{1}{v_1-w_1}\int_{w_1}^{v_1}\Phi_1(\eta)\,d\eta \leqslant \frac{\Phi_1(w_1)+\Phi_1(v_1)}{2} \tag{1}$$

It is known as the Hermite-Hadamard inequality for a convex function. That is, in a set of real numbers, if a function is convex, its weighted average value at the endpoints will be equal to or greater than its value at the middle of any interval. See [15]-[18] for more improvement, extension and generalization about this Inequality (1).

let $\Phi_1 : \xi \subset \mathbb{R} \to \mathbb{R}$ be convex function and $w_1, v_1, \in \xi$ with $0 \leq w_1 < v_1$ such that $\Phi_1 \in L[w_1, v_1]$. If Φ_1 is convex on $L[w_1, v_1]$, then the inequality

$$\Phi_1\left(\frac{w_1+v_1}{2}\right) \leqslant \frac{\Gamma(\nu+1)}{2(v_1-w_1)}\left(\mathbb{I}^{\nu}_{v_1^-}\Phi_1(w_1) + \mathbb{I}^{\nu}_{w_1^+}\Phi_1(v_1)\right) \leqslant \frac{\Phi_1(w_1)+\Phi_1(v_1)}{2} \tag{2}$$

with $\nu > 0$, is known as the fractional Hermite-Hadamard inequality, where $\mathbb{I}^{+}_{w_1}$ and $\mathbb{I}^{-}_{v_1}$ stand for the right-sided and the left-sided Riemann-Liouville fractional integrals of the order ν. It is noteworthy that the fractional Hermite-Hadamard inequality simplifies to the classical Hermite-Hadamard inequality when $\nu = 1$ in Inequality (2).

Researchers can create more sophisticated methods for examining and enhancing systems that exhibit fractional dynamic behavior by connecting fractional Hermite-Hadamard inequalities to the broader framework of fractional integral operators and generalized convexity. The interaction of these ideas enables the creation of new, more accurate mathematical inequalities that can be used in a variety of intricate systems. This integrated method provides deeper insights into the mathematical processes controlling fractional calculus and its applications in science and engineering, opening up new avenues for research; for more details, see references [19]-[21].

To the best of our knowledge, this paper provides a novel and in-depth analysis of exponentially (s, m)-convex functions about Katugampola fractional integrals. Antczak [22] introduced the notion of exponentially convex functions, which can be seen as a substantial generalization of convex functions. Exponentially convex functions play a significant role in diverse areas, including mathematical programming, information geometry, big data analysis, machine learning, statistics, sequential prediction, and stochastic optimization [23], [24]. Moreover, Rashid *et al.* [25] established some trapezoid-type inequalities for generalized fractional integrals and related inequalities via exponentially convex functions. Rashid *et al.* [26] derived a new integral identity involving Riemann–Liouville fractional integrals and obtained new fractional bounds for the functions having the exponential convexity property. Rashid *et al.* [27] introduced some new generalizations for exponentially s-convex functions and inequalities via fractional operators. Recognizing the significance of fractional integrals in numerous areas of pure and applied science, researchers have expanded their concept in various ways, which leads to the development of new integral inequalities for these generalized fractional integrals.

Recently, Khan *et al*. [28] established some new developments of Hermite–Hadamard-type inequalities via s-convexity and fractional integrals. Inspired by the mentioned research effort and notions, we study the concept of exponentially (s, m)-convexity to derive inequalities of fractional Hermite-Hadamard type exponentially (s, m)-convex functions and some generalizations associated with these inequalities.

The primary objective of this study is to develop Hermite-Hadamard type inequalities to Katugampola fractional integrals. To this end, we first introduce a new integral identity on which we base the establishment of several Hermite-Hadamard type inequalities for functions with extended (s, m)-convex first-order derivatives. Subsequently, we present an example that includes graphical representations to confirm the validity of our results. Hermite-Hadamard inequalities are powerful tools for establishing bounds for symmetric expressions. When paired with (s, m)-convex functions, these inequalities become more versatile and yield sharper estimates. The proposed work is structured as follows: In section 2, we give some essential definitions, important theorems, and a generalized lemma that are required for our major result. In section 3, we state and prove our key results utilizing the generalized lemma and theorems, as well as deriving several new corollaries and giving some important remarks. In section 4, a conclusion is drawn.

2. RESEARCH METHOD

In this section, we collect some notations, basic definitions and essential results required in the sequel of the paper.

Definition 1. [29] A function $\Phi_1 : I \to \mathbb{R}$ is said to be a convex function if

$$\Phi_1 (\eta w_1 + (1-\eta) v_1) \leqslant \eta \Phi_1 (w_1) + (1-\eta) \Phi_1 (v_1)$$

holds for all $w_1, v_1 \in I$ and $\eta \in [0, 1]$.

Definition 2. [30] $\Phi_1 : [0, b] \to \mathbb{R}$ is said to be a m-convex function if

$$\Phi_1 (\eta w_1 + m (1-\eta) v_1) \leq \eta \Phi_1 (w_1) + m (1-\nu) \Phi_1 (v_1)$$

holds for all $w_1, v_1 \in [0, b]$, $\eta \in [0, 1]$, and $m \in (0, 1]$. In [30], Toader introduced the above concept of an m-convex function.

Definition 3. [31] $\Phi_1 : [0, b] \to \mathbb{R}$ is said to be a s-convex function if

$$\Phi_1 (\eta w_1 + (1-\eta) v_1) \leq \eta^s \Phi_1 (w_1) + (1-\eta)^s \Phi_1 (v_1)$$

holds for all $w_1, v_1 \in [0, b]$, $\eta \in [0, 1]$, and $s \in (0, 1]$.

Definition 4. [32] A function $\Phi_1 : [0, \eta] \to \mathbb{R}$ is said to be an (s, m)-convex function, where $(s, m) \in [0, 1]^2$ and $\eta > 0$, iff $\forall\ w_1, v_1 \in [0, \eta]$ and $\eta \in [0, 1]$ if

$$\Phi_1 \bigg(\eta w_1 + m(1-\eta) v_1 \bigg) \leq \eta^s \Phi_1(w_1) + m(1-\eta)^s \Phi_1(v_1)$$

Definition 5. A positive real-valued function $\Phi_1 : I \subseteq \mathbb{R} \to (0, \infty)$ is said to be exponentially convex on K, if

$$e^{\Phi_1(\eta w_1 + (1-\eta) v_1)} \leq \eta e^{\Phi_1(k_1)} + (1-\eta) e^{\Phi_1(v_1)}$$

Exponentially convex functions are utilized in statistical learning, sequential prediction, and stochastic optimization.

Definition 6. [33] A function $\Phi_1 : I \to \mathbb{R}$ is said to be a exponentially s- convex function in the first sence, if the following inequality holds:

$$e^{\Phi_1(w_1 \eta + v_1(1-\eta))} \leq \eta^s e^{\Phi_1(w_1)} + (1-\eta)^s e^{\Phi_1(v_1)}, \forall\ s \in [0, 1],\ w_1, v_1 \in I,\ \eta \in [0, 1].$$

For $\eta = \frac{1}{2}$, we get

$$e^{\Phi_1(\frac{w_1+v_1}{2})} \leq \frac{e^{\Phi_1(w_1)+\Phi_1(v_1)}}{2^s}, \; \forall \, w_1, v_1 \in I,$$

which is called exponentially Jensen-convex function.

Definition 7. Let $s \in [0,1]$ and $I \subseteq [0,\infty)$. A function $\Phi : I \to \mathbb{R}$ is said to be exponentially $(s,m)-$convex function in the second sense if

$$\Phi_1(\eta w_1 + m(1-\eta)v_2) \leq \frac{\eta^s \Phi_1(w_1)}{e^{\xi_1 w_1}} + \frac{m(1-\eta)^s \Phi_1(w_1)}{e^{\xi_1 v_1}},$$

holds for all $w_1, v_1 \in I, \; m \in [0,1] \; and \; \xi_1 \in \mathbb{R}$.

Definition 8. [33] Let $\Phi_1 \in \pounds_{w_1}^{v_1}(w_1, v_1)$. The left and right-sided Katugampola fractional integrals of order $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0 \; and \; \sigma' > 0$ are defined by

$${}^{\sigma'}\mathbb{I}^{\nu}_{w_1^+}\Phi_1(x) = \frac{\sigma'^{(1-\nu)}}{\Gamma(\mu)} \int_{w_1}^{x} \frac{\eta^{\sigma'-1}\Phi_1(\eta)}{(x^{\sigma'} - \eta^{\sigma'})^{1-\nu}} d\eta, \; x > w_1,$$

$${}^{\sigma'}\mathbb{I}^{\nu}_{v_1^-}\Phi_1(x) = \frac{\sigma'^{(1-\nu)}}{\Gamma(\nu)} \int_{x}^{v_1} \frac{\eta^{\sigma'-1}\Phi_1(\eta)}{(\eta^{\sigma'} - x^{\sigma'})^{1-\nu}} d\eta, \; v_1 > x.$$

Where $\pounds_{w_1}^{v_1}(w_1, v_1)(w_1 \in \mathbb{R} \; and \; 1 \leq v_1 \leq \infty)$denotes the space of all complex-valued Lebesgue measurable functions Φ_1 for which $||\Phi_1||_{\pounds_{w_1}^{v_1}} < \infty$ and the norm is defined by

$$||\Phi_1||_{\pounds_{w_1}^{v_1}} = \left(\int_{w_1}^{v_1} |\eta^{w_1} g(\eta)|^{v_1} \right)^{\frac{1}{v_1}} \quad for \;\; 1 \leq v_1 < \infty$$

and for r=∞

$$||\Phi_1||_{\pounds_{w_1}^{\infty}} = ess \sup_{w_1 \leq \eta \leq v_1} |\eta^{w_1}\Phi_1(\eta)|.$$

Definition 9. [34] Let $\Phi_1 \in L'[w_1, v_1]$. The fractional Riemann-Liouville integrals $\jmath^{\nu}_{w_1^+}\Phi_1$ and $\jmath^{\nu}_{v_1^-}\Phi_1$ of order ν are defined by

$$\jmath^{\nu}_{w_1^+}\Phi_1(x) = \frac{1}{\Gamma(\nu)} \int_{w_1}^{x} (x-\eta)^{\nu-1}\Phi_1(\eta) d\eta, x > w_1,$$

$$\jmath^{\nu}_{v_1^-} g_2(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{v_1} (\eta - x)^{\nu-1}\Phi_1(\eta) d\eta, v_1 > x,$$

where $\Gamma(\nu)$ is a gamma function.

Definition 10. [35] The left and right-sided Hadamard's fractional integral operators of order $\nu > 0$ are defined by

$$H^{\nu}_{w_1^+}\Phi_1(x) = \frac{1}{\Gamma(\nu)} \int_{w_1}^{x} (ln\, x - ln\, \eta)^{\nu-1} \frac{\Phi_1(\eta)}{\eta} d\eta, x > w_1,$$

$$H^{\nu}_{v_1^-} g_2(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{v_1} (ln\, \eta - ln\, x)^{\nu-1} \frac{\Phi_1(\eta)}{\eta} d\eta, v_1 > x.$$

Theorem 1. Let $\nu > 0$ and $\sigma' > 0$. Then, for $x > w_1$,
1. $lim^{\sigma'}_{\sigma' \to 1} \mathcal{J}^{\nu}_{w_1^+} \Phi_1(x) = I^{\nu}_{w_1^-} \Phi_1(x)$,
2. $\lim_{\sigma' \to 0^+} \mathcal{J}^{\nu}_{w_1^+} \Phi_1(x) = H^{\nu}_{w_1^-} \Phi_1(x)$

We recall the special functions that are known as Gamma function and beta function, respectively.

$$\Gamma(x) = \int_0^{\infty} e^{-\eta} \eta^{x-1} d\eta,$$

$$\mathbb{B}(x, y) = \int_0^1 \eta^{x-1} (1-\eta)^{(y-1)} d\eta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,$$

where $\mathbb{B}(.,.)$ denotes beta as a special function.

The Katugampola fractional integral is a powerful fractional calculus tool that unifies both the Riemann–Liouville and Hadamard fractional integrals. This work presents a Hermite-Hadamard type inequality that incorporates the properties of exponentially (s, m) convex functions to extend and strengthen classical results.

3. RESULTS AND DISCUSSION

In this section, we first establish an identity involving the Katugampola fractional integral. Then we establish an integral inequality involving beta function. We begin with the following lemma which is used to explore integral inequality.

Lemma 2. Let $\sigma' > 0,\ \nu > 0$ and $\Phi_1 : [w_1^{\sigma'}, v_1^{\sigma'}] \to \mathbb{R}$ be differentiable function on $[w_1^{\sigma'}, v_1^{\sigma'}]$ with $0 \leq w_1^{\sigma'} < v_1^{\sigma'}$, and $\Phi_1{}' \in L^1[w_1^{\sigma'}, v_1^{\sigma'}]$. Then the following equality holds:

$$\begin{aligned}
&\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{2} \left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-} \Phi_1(m^{\sigma'} w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^{\nu}} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+} \Phi_1(m^{\sigma'} v_1^{\sigma'})}{(m^{\sigma'} v_1^{\sigma'} - k_1^{\sigma'})^{\nu}} \right] \\
&= \frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'}))}{2} \int_0^1 \eta^{\sigma'\nu} \eta^{\sigma'-1} \Phi_1'(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) w_1^{\sigma'}) d\eta \\
&\quad + \frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'}))}{2} \int_0^1 \eta^{\sigma'\nu} \eta^{\sigma'-1} \Phi_1'(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) v_1^{\sigma'}) d\eta.
\end{aligned}$$

Proof

$$Let\ H_1 = \int_0^1 \eta^{\sigma'\nu} \eta^{\sigma'-1} \Phi_1'(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) w_1^{\sigma'}) d\eta,$$

$$and\ H_2 = \int_0^1 \eta^{\sigma'\nu} \eta^{\sigma'-1} \Phi_1'(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) v_1^{\sigma'}) d\eta.$$

$Integreating\ by\ parts\ H_1,\ we\ get$

$$\begin{aligned}
H_1 &= \left[\frac{\eta^{\sigma'\nu} \Phi_1(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) w_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})} \right]_0^1 \\
&\quad - \frac{\nu}{\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})} \int_0^1 \eta^{\sigma'\nu-1} \Phi_1(\eta^{\sigma'} k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'}) w_1^{\sigma'}) d\eta \\
&= \frac{\Phi_1(k_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})} - \frac{\nu}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^{\nu+1}} \int_{mw_1}^{k_1} (u^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^{\nu-1} u^{\nu-1} \Phi_1(u^{\sigma'}) du \\
&= \frac{\Phi_1(k_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})} - \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^{\nu+1}} {}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-} \Phi_1(m^{\sigma'} w_1^{\sigma'}).
\end{aligned}$$

$$Now,\ (\sigma'(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'}))H_1 = \Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^{\nu}} {}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-} \Phi_1(m^{\sigma'} w_1^{\sigma'}). \tag{3}$$

Similarly, we have

$$
\begin{aligned}
H_2 &= \int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta \\
&= \left[\frac{\eta^{\sigma'\nu}\Phi_1(\eta^{\sigma'k_1^{\sigma'}} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}\right]_0^1 \\
&\quad - \frac{\nu}{\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}\int_0^1 \eta^{\sigma'\nu-1}\Phi_1(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta \\
&= \frac{\Phi_1(k_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})} + \frac{\nu}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu+1}}\int_{k_1}^{mv_1}(m^{\sigma'}v_1^{\sigma'} - u^{\sigma'})^{\nu-1}u^{\nu-1}\Phi_1(u^{\sigma'})du \\
&= \frac{\Phi_1(k_1^{\sigma'})}{\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})} + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu+1}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'}). \\
&Now,\ (\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'}))H_2 = \Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'}).
\end{aligned} \tag{4}
$$

Now adding (3) and (4), we get

$$
\begin{aligned}
&\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\right] \\
&= \frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})d\eta \\
&\quad + \frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta.
\end{aligned}
$$

Thus the proof is completed.

Remark 1. If we put $m = \sigma' = 1$ in Lemma 2, then we get Lemma (2) of [28].

Theorem 3. Let $\sigma' > 0,\ \nu > 0$ and $\Phi_1 : [w_1^{\sigma'}, v_1^{\sigma'}] \to \mathbb{R}$ be a positive function with $0 \le w_1^{\sigma'} < k_1^{\sigma'} < v_1^{\sigma'}$, and ${\Phi_1}' \in L^1[w_1^{\sigma'}, v_1^{\sigma'}]$. If Φ_1 is exponentially (s, m)-convex function on $[w_1^{\sigma'}, v_1^{\sigma'}]$, then the following inequality for fractional integrals holds:

$$
\begin{aligned}
&\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'}) \\
&\le \frac{\nu}{\sigma'}\left[2\mathbb{B}(\nu+s,1)\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}} + \mathbb{B}(\nu, s+1)m^{\sigma'}\left(\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}} + \frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right)\right],
\end{aligned}
$$

for all $m, s \in (0, 1]$ and $\xi_1 \in \mathbb{R}$.

Proof Applying exponentially (s, m) convexity of Φ_1

$$
\begin{aligned}
&\Phi_1(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'}) + \Phi_1(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'}) \\
&\le \eta^{\sigma's}\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}} + m^{\sigma'}(1-\eta^{\sigma'})^s\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}} + \eta^{\sigma's}\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}} + m^{\sigma'}(1-\eta^{\sigma'})^s\frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}
\end{aligned} \tag{5}
$$

Multiply both sides of (5) by $\eta^{\sigma'\nu-1}$ and integrate w.r.t η over [0,1]

$$\begin{aligned}
&\int_0^1 \eta^{\sigma'\nu-1}\Phi_1(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})d\eta\\
&+\int_0^1 \eta^{\sigma'\nu-1}\Phi_1(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta\\
&\leq 2\int_0^1 \eta^{\sigma'\nu-1}\eta^{\sigma' s}\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}d\eta+\int_0^1 \eta^{\sigma'\nu-1}m^{\sigma'}(1-\eta^{\sigma'})^s\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}d\eta\\
&\quad+\int_0^1 \eta^{\sigma'\nu-1}m^{\sigma'}(1-\eta^{\sigma'})^s\frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}d\eta\\
&=\frac{2}{\sigma'(\nu+s)}\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}+\frac{m^{\sigma'}}{\sigma'}\left(\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}+\frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right)\mathbb{B}(\nu,s+1).
\end{aligned} \tag{6}$$

Now consider $t^{\sigma'}=(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})$ in first term of L.H.S of (6) and $r^{\sigma'}=(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})$ in second term of L.H.S of (6).

$$\begin{aligned}
&\int_{mw}^{k_1}\left(\frac{t^{\sigma'}-m^{\sigma'}w_1^{\sigma'}}{k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'}}\right)^{\nu-1}t^{\sigma'-1}\Phi_1(t^{\sigma'})\frac{dt}{k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'}}\\
&\quad+\int_{k_1}^{mw}\left(\frac{m^{\sigma'}v_1^{\sigma'}-r^{\sigma'}}{m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'}}\right)^{\nu-1}r^{\sigma'-1}\Phi_1(r^{\sigma'})\frac{dr}{m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'}}\\
&\leq\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\frac{2}{\sigma'(\nu+s)}+\frac{m^{\sigma'}}{\sigma'}\left(\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}+\frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right)\mathbb{B}(\nu,s+1).\\
&\textit{By multiplying both sides by } \nu, \textit{ we get}\\
&\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})+\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})\\
&\leq\frac{\nu}{\sigma'}\left[2\mathbb{B}(\nu+s,1)\frac{\Phi_1(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}+m^{\sigma'}\left(\frac{\Phi_1(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}+\frac{\Phi_1(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right)\mathbb{B}(\nu,s+1)\right].
\end{aligned}$$

Thus the proof is completed.

Remark 2. If we put $m=\sigma'=1$ and $\xi_1=0$ in Theorem 3, then we get Theorem (4) of [28].

In [28] at Theorem (4), researchers use the concept of $(R-L)$ fractional Hermite-Hadamard integral inequality by using $s-$convexity. In Theorem 3, we extended the result using the Katugampola fractional Hermite-Hadamard integral inequality with exponential $(s,m)-$convexity, which becomes more versatile and yields sharper estimates. For better understanding, we provide an example illustrating our theoretical findings, supported by a graphical representation.

Example 1. If we choose $s=m=1\in(0,1]$, $\xi_1=0\in\mathbb{R}$, $\sigma'=1$, $\eta=\frac{1}{2}\in[0,1]$, $k_1=\frac{5}{2}$, and $v_1=3$ in Theorem 3, then $\Phi_1(t)=t^4$ is an exponentially (s,m)-convex, as Theorem 3 satisfying the following estimation:

$$\frac{195.31-2w_1^5}{25-10w_1}+58.13\leq 39.06+\frac{w_1^4+81}{2} \tag{7}$$

We have shown the graphical representation of Inequality (7) using MATLAB R2019a software.

Remark 3. From Figure 1, we observe that in the Inequality (7), the left hand side gives more accurate estimate than the right hand side of Theorem 3 graphically. In Figure 1, the vertical axis is represented by w, and the horizontal axis is represented by w_1.

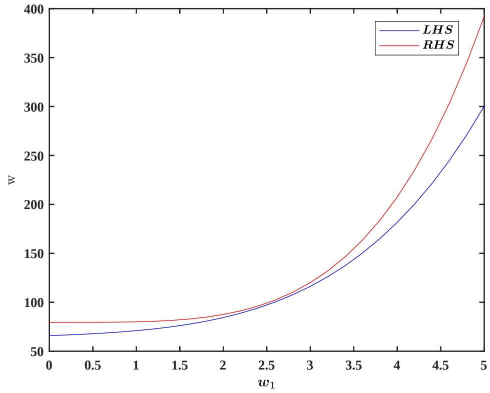

Figure 1. Graphical description of Inequality (7), which provides better understanding and validity of Theorem 3

Corollary 1. If we choose $s = 1$ in Theorem 3, we deduc

$$\left|\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})\right| \\ \leq \frac{\nu}{\sigma'}\left[2\mathbb{B}(\nu+1,1)\Phi_1(k_1^{\sigma'})e^{-(\xi_1 k_1^{\sigma'})} + \mathbb{B}(\nu,2)m^{\sigma'}\left(\Phi_1(w_1^{\sigma'})e^{-(\xi_1 w_1^{\sigma'})} + \Phi_1(v_1^{\sigma'})e^{-(\xi_1 v_1^{\sigma'})}\right)\right]$$

Corollary 2. If we choose $s = 0$ in Theorem 3, we deduce

$$\left|\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})\right| \\ \leq \frac{1}{\sigma'}\left[2\Phi_1(k_1^{\sigma'})e^{-(\xi_1 k_1^{\sigma'})} + m^{\sigma'}\left(\Phi_1(w_1^{\sigma'})e^{-(\xi_1 w_1^{\sigma'})} + \Phi_1(v_1^{\sigma'})e^{-(\xi_1 v_1^{\sigma'})}\right)\right]$$

Corollary 3. If we choose $\xi_1 = 0$ in Theorem 3, we deduce

$$\left|\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})\right| \\ \leq \frac{\nu}{\sigma'}\left[2\mathbb{B}(\nu+s,1)\Phi_1(k_1^{\sigma'}) + \mathbb{B}(\nu,s+1)m^{\sigma'}\left(\Phi_1(w_1^{\sigma'}) + \Phi_1(v_1^{\sigma'})\right)\right],$$

Corollary 4. If we choose $m = 1$ in Theorem 3, we deduce

$$\left|\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - w_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\,{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(v_1^{\sigma'})\right| \\ \leq \frac{\nu}{\sigma'}\left[2\mathbb{B}(\nu+s,1)\Phi_1(k_1^{\sigma'})e^{-(\xi_1 k_1^{\sigma'})} + \mathbb{B}(\nu,s+1)\left(\Phi_1(w_1^{\sigma'})e^{-(\xi_1 w_1^{\sigma'})} + \Phi_1(v_1^{\sigma'})e^{-(\xi_1 v_1^{\sigma'})}\right)\right]$$

Theorem 4. Let $\sigma' > 0$, $\nu > 0$ and $\Phi_1 : I \to \mathbb{R}$ be a differentiable mapping on I^o, and $w_1^{\sigma'}, v_1^{\sigma'} \in I^o$ with $w_1^{\sigma'} < k_1^{\sigma'} < v_1^{\sigma'}$ such that ${\Phi_1}' \in L^1[w_1^{\sigma'}, v_1^{\sigma'}]$. If Φ_1' is exponentially (s, m)-convex function on $[w_1^{\sigma'}, v_1^{\sigma'}]$, then the following inequality for fractional integrals holds:

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\right]\right| \\ \leq \frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right] \\ + \frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right],$$

holds $\forall\, m, s \in (0, 1]$ and $\xi_1 \in \mathbb{R}$.

proof By taking absolute value in Lemma 2, we deduce

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right|$$
$$\le \frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})d\eta$$
$$+\frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'}+m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta,$$

since $|\Phi_1'|$ $is\ exponentially\ (s,m)$ convexity

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right|$$
$$\le \frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'}))}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\int_0^1 \eta^{\sigma'(\nu+s+1)-1}d\eta\right.$$
$$\left.+m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\int_0^1 \eta^{\sigma'(\nu+1)-1}(1-\eta^{\sigma'})^s d\nu\right]$$
$$+\frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'}))}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\int_0^1 \eta^{\sigma'(\nu+s+1)-1}d\eta\right.$$
$$\left.+m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\int_0^1 \eta^{\sigma'(\nu+1)-1}(1-\eta^{\sigma'})^s d\nu\right]$$
$$=\frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1)+m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right]$$
$$+\frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1)+m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right].$$

Thus the proof is completed.

Remark 4. If we put $m = \sigma' = 1$ and $\xi_1 = 0$ in Theorem 4, then we get Theorem (5) of [28].

In [28] at Theorem (5), researchers use the concept of $(R-L)$ fractional Hermite-Hadamard integral inequality by using $s-$convexity and the property of modulus. In Theorem 4, we extended the result using Katugampola fractional Hermite-Hadamard integral inequality with exponential $(s,m)-$convexity and the property of modulus, which becomes more versatile and yields sharper estimates. For better understanding, we provide an example illustrating our theoretical findings, supported by a graphical representation.

Example 2. If we choose $s = m = 1 \in (0,1]$, $\xi_1 = 0 \in \mathbb{R}$, $\sigma' = 1$, $\eta = \frac{1}{2} \in [0,1]$, $k_1 = \frac{5}{2}$, and $v_1 = 3$ in Theorem 4, then $\Phi_1(t) = t^4$ is an exponentially (s,m)-convex, as Theorem 4 satisfying the following estimation:

$$10 - \frac{195.31 - 2w_1^5}{50 - 20w_1} \le \frac{5 - 2w_1}{4}\left[20.83 + \frac{2}{3}w_1^3\right] - 9.70. \tag{8}$$

We have shown the graphical representation of Inequality (8) using MATLAB R2019a software.

Remark 5. From Figure 2, we observe that in the Inequality (8), the left hand side gives more accurate estimate than the right hand side of Theorem 4 graphically. In Figure 2, the vertical axis is represented by w, and the horizontal axis is represented by w_1.

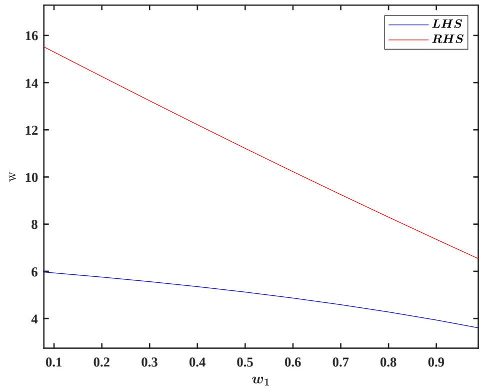

Figure 2. Graphical description of Inequality (8), which provides better understanding and validity of Theorem 4

Corollary 5. If we choose $s = 1$ in Theorem 4, we deduce

$$\begin{aligned}&\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|\\ &\leq \frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+2),1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,2)\right]\\ &\quad + \frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+2),1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,2)\right].\end{aligned}$$

Corollary 6. If we choose $s = 0$ in Theorem 4, we deduce

$$\begin{aligned}&\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|\\ &\leq \frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+1),1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,1)\right]\\ &\quad + \frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+1),1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,1)\right].\end{aligned}$$

Corollary 7. If we choose $\xi_1 = 0$ in Theorem 4, we deduce

$$\begin{aligned}&\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\\ &\leq \frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|\mathbb{B}((\nu+s+1),1) + m^{\sigma'}\left|\Phi_1'(w_1^{\sigma'})\right|\mathbb{B}(\nu+1,s+1)\right]\\ &\quad + \frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|\mathbb{B}((\nu+s+1),1) + m^{\sigma'}\left|\Phi_1'(v_1^{\sigma'})\right|\mathbb{B}(\nu+1,s+1)\right].\end{aligned}$$

Corollary 8. If we choose $m = 1$ in Theorem 4, we deduce

$$\begin{aligned}&\left|\frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(k_1^{\sigma'} - w_1^{\sigma'})^\nu}{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(w_1^{\sigma'}) + \frac{(\sigma')^{\nu-1}\Gamma(\nu+1)}{(v_1^{\sigma'} - k_1^{\sigma'})^\nu}{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(v_1^{\sigma'})\right|\\ &\leq \frac{(k_1^{\sigma'} - w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1) + \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right]\\ &\quad + \frac{(k_1^{\sigma'} - v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|\mathbb{B}((\nu+s+1),1) + \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|\mathbb{B}(\nu+1,s+1)\right].\end{aligned}$$

Theorem 5. Let $\sigma' > 0,\ \nu > 0$ and $\Phi_1 : I \to \mathbb{R}$ be a differentiable mapping on I^o, and $w_1^{\sigma'}, v_1^{\sigma'} \in I^o$ with $w_1^{\sigma'} < k_1^{\sigma'} < v_1^{\sigma'}$ such that ${\Phi_1}' \in L^1[w_1^{\sigma'}, v_1^{\sigma'}]$. If $|\Phi_1'|^{\mu_2}$ is exponentially (s,m)-convex function on $[w_1^{\sigma'}, v_1^{\sigma'}]$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
&\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right| \\
&\leq \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}} \\
&\times \left\{\frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right. \\
&\left. + \frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right\}, \qquad (9)
\end{aligned}$$

holds $\forall\ m, s \in (0,1]$, with $\mu_2 > 1$ and $\xi_1 \in \mathbb{R}$.

proof By taking absolute value in Lemma 2, we deduce

$$\begin{aligned}
&\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right| \\
&\leq \frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})d\eta \\
&\quad + \frac{(\sigma'(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta \\
&= \frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\int_0^1 \varepsilon^\nu\Phi_1'(\varepsilon k_1^{\sigma'} + m^{\sigma'}(1-\varepsilon)w_1^{\sigma'})d\varepsilon \\
&\quad + \frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\int_0^1 \varepsilon^\nu\Phi_1'(\varepsilon k_1^{\sigma'} + m^{\sigma'}(1-\varepsilon)v_1^{\sigma'})d\varepsilon. \qquad (10)
\end{aligned}$$

By applying Hölder ineqality and using exponentially (s,m)-convexity

$$\begin{aligned}
&\leq \frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left(\int_0^1\varepsilon^\nu d\varepsilon\right)^{1-\frac{1}{\mu_2}} \\
&\quad\times\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\int_0^1\varepsilon^{\nu+s}d\varepsilon + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\int_0^1\varepsilon^\nu(1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}} \\
&+ \frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left(\int_0^1\varepsilon^\nu d\varepsilon\right)^{1-\frac{1}{\mu_2}} \\
&\quad\times\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\int_0^1\varepsilon^{\nu+s}d\varepsilon + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\int_0^1\varepsilon^\nu(1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}} \\
&= \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}} \\
&\times \left\{\frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right. \\
&\left. + \frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right\}.
\end{aligned}$$

Thus the proof is completed.

Remark 6. If we put $m = \sigma' = 1$ and $\xi_1 = 0$ in Theorem 5, then we get Theorem (6) of [28].

Remark 7. By applying the Power-mean integral Inequality in (10) then we also get the Inequality (9).

In [28] at Theorem (6), researchers use the concept of $(R-L)$ fractional Hermite-Hadamard integral inequality by using $s-$convexity and well-known Hölder inequality. In Theorem 5, we extended the result using Katugampola fractional Hermite-Hadamard integral inequality with exponential $(s,m)-$convexity and well-known Hölder inequality, which becomes more versatile and yields sharper estimates. For better understanding, we provide an example illustrating our theoretical findings, supported by a graphical representation (see Figure 3).

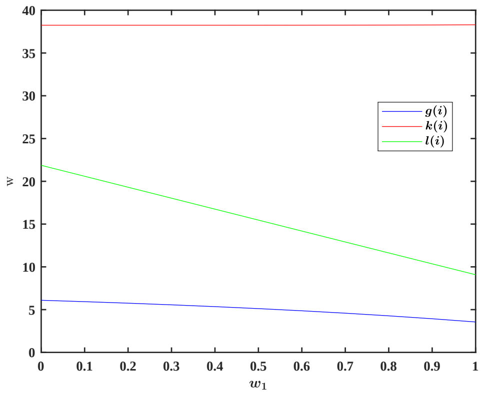

Figure 3. Graphical comperision of Inequalites (11) and (12)

Remark 8. From Figure 3, we observe that in the Inequality (11) and (12), the left hand side gives more accurate estimate than the right hand side of Theorem 5 and Theorem 6 graphically. In Figure 3, the vertical axis is represented by w, and the horizontal axis is represented by w_1.

Corollary 9. If we choose $s = 1$ in Theorem 5, we deduce

$$
\begin{aligned}
&.\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right| \\
&\leq \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}} \\
&\times\left\{\frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+2,1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,2)\right]^{\frac{1}{\mu_2}}\right. \\
&\left.+\frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+2,1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,2)\right]^{\frac{1}{\mu_2}}\right\}.
\end{aligned}
$$

Corollary 10. If we choose $s = 0$ in Theorem 5, we deduce

$$
\begin{aligned}
&\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'}-k_1^{\sigma'})^\nu}\right]\right| \\
&\leq \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}} \\
&\times\left\{\frac{(k_1^{\sigma'}-m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,1)\right]^{\frac{1}{\mu_2}}\right. \\
&\left.+\frac{(k_1^{\sigma'}-m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,1) + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,1)\right]^{\frac{1}{\mu_2}}\right\}.
\end{aligned}
$$

Corollary 11. *If we choose $\xi_1 = 0$ in Theorem 5, we deduce*

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}}$$
$$\times\left\{\frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\Phi_1'(w_1^{\sigma'})\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right.$$
$$\left.+\frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + m^{\sigma'}\left|\Phi_1'(v_1^{\sigma'})\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right\}.$$

Corollary 12. If we choose $m = 1$ in Theorem 5, we deduce

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(w_1^{\sigma'})}{(k_1^{\sigma'} - w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(v_1^{\sigma'})}{(v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \left(\frac{1}{\nu+1}\right)^{1-\frac{1}{\mu_2}}$$
$$\times\left\{\frac{(k_1^{\sigma'} - w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right.$$
$$\left.+\frac{(k_1^{\sigma'} - v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+s+1,1) + \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\mathbb{B}(\nu+1,s+1)\right]^{\frac{1}{\mu_2}}\right\}.$$

Theorem 6. Let $\sigma' > 0,\ \nu > 0$ and $\Phi_1 : I \to \mathbb{R}$ be a differentiable mapping on I^o, and $w_1^{\sigma'}, v_1^{\sigma'} \in I^o$ with $w_1^{\sigma'} < k_1^{\sigma'} < v_1^{\sigma'}$ such that $\Phi_1{}' \in L^1[w_1^{\sigma'}, v_1^{\sigma'}]$. If $|\Phi_1'|^{\mu_2}$ is exponentially (s,m)-convex function on $[w_1^{\sigma'}, v_1^{\sigma'}]$, then the following inequality for fractional integrals holds:

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \left(\frac{1}{\nu\mu_1+1}\right)^{\frac{1}{\mu_1}}\left(\frac{1}{s+1}\right)^{\frac{1}{\mu_2}}\left\{\frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'}\left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right.$$
$$\left.+\frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'}\left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right\},$$

holds $\forall\, m, s \in (0,1]$, with $\mu_2 > 1$ and $\xi_1 \in \mathbb{R}$.

proof By taking absolute value in Lemma 2, we deduce

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^\nu\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})w_1^{\sigma'})d\eta$$
$$+\frac{(\sigma'(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'}))}{2}\int_0^1 \eta^{\sigma'\nu}\eta^{\sigma'-1}\Phi_1'(\eta^{\sigma'}k_1^{\sigma'} + m^{\sigma'}(1-\eta^{\sigma'})v_1^{\sigma'})d\eta$$
$$= \frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\int_0^1 \varepsilon^\nu\Phi_1'(\varepsilon k_1^{\sigma'} + m^{\sigma'}(1-\varepsilon)w_1^{\sigma'})d\varepsilon$$
$$+\frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\int_0^1 \varepsilon^\nu\Phi_1'(\varepsilon k_1^{\sigma'} + m^{\sigma'}(1-\varepsilon)v_1^{\sigma'})d\varepsilon.$$

By applying Hölder ineqality and using exponentially (s, m)-convexity, we have

$$\leq \frac{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})}{2}$$
$$\times \left(\int_0^1 \varepsilon^{\nu\mu_1} d\varepsilon\right)^{\frac{1}{\mu_1}} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 \varepsilon^s d\varepsilon + m^{\sigma'} \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 (1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}}$$
$$+ \frac{(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'})}{2}$$
$$\times \left(\int_0^1 \varepsilon^{\nu\mu_1} d\varepsilon\right)^{\frac{1}{\mu_1}} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 \varepsilon^s d\varepsilon + m^{\sigma'} \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 (1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}}$$
$$= \frac{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})}{2}$$
$$\times \left(\frac{1}{\nu\mu_1 + 1}\right)^{\frac{1}{\mu_1}} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 \varepsilon^s d\varepsilon + m^{\sigma'} \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 (1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}}$$
$$+ \frac{(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'})}{2}$$
$$\times \left(\frac{1}{\nu\mu_1 + 1}\right)^{\frac{1}{\mu_1}} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 \varepsilon^s d\varepsilon + m^{\sigma'} \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2} \int_0^1 (1-\varepsilon)^s d\varepsilon\right]^{\frac{1}{\mu_2}}$$
$$= \left(\frac{1}{\nu\mu_1 + 1}\right)^{\frac{1}{\mu_1}} \left(\frac{1}{s+1}\right)^{\frac{1}{\mu_2}} \left\{ \frac{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right.$$
$$\left. + \frac{(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right\}.$$

Thus the proof is completed.

Remark 9. If we put $m = \sigma' = 1$ and $\xi_1 = 0$ in Theorem 6, then we get Theorem (8) of [28].

In [28] at Theorem (8), researchers use the concept of $(R - L)$ fractional Hermite-Hadamard integral inequality by using $s-$convexity and well-known Hölder inequality. In Theorem 6, we extended the result using Katugampola fractional Hermite-Hadamard integral inequality with exponential $(s, m)-$convexity and well-known Hölder inequality, which becomes more versatile and yields sharper estimates. For better understanding, we provide an example illustrating our theoretical findings, supported by a graphical representation (see Figure 3).

Corollary 13. If we choose $s = 1$ in Theorem 6, we deduce

$$\left|\Phi(k_1^{\sigma'}) - \frac{(\sigma')^\nu \Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi(m^{\sigma'} w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi(m^{\sigma'} v_1^{\sigma'})}{(m^{\sigma'} v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \left(\frac{1}{\nu\mu_1 + 1}\right)^{\frac{1}{\mu_1}} \left(\frac{1}{2}\right)^{\frac{1}{\mu_2}} \left\{ \frac{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right.$$
$$\left. + \frac{(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right\}.$$

Corollary 14. If we choose $s = 0$ in Theorem 6, we deduce

$$\left|\Phi(k_1^{\sigma'}) - \frac{(\sigma')^\nu \Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^-}\Phi(m^{\sigma'} w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})^\nu} + \frac{{}^{\sigma'}\mathbb{I}^\nu_{k_1^+}\Phi(m^{\sigma'} v_1^{\sigma'})}{(m^{\sigma'} v_1^{\sigma'} - k_1^{\sigma'})^\nu}\right]\right|$$
$$\leq \left(\frac{1}{\nu\mu_1 + 1}\right)^{\frac{1}{\mu_1}} \left\{ \frac{(k_1^{\sigma'} - m^{\sigma'} w_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right.$$
$$\left. + \frac{(k_1^{\sigma'} - m^{\sigma'} v_1^{\sigma'})}{2} \left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + m^{\sigma'} \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}} \right\}.$$

Corollary 15. If we choose $\xi_1 = 0$ in Theorem 6, we deduce

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(m^{\sigma'}w_1^{\sigma'})}{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})^{\nu}} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(m^{\sigma'}v_1^{\sigma'})}{(m^{\sigma'}v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\right]\right|$$
$$\leq \left(\frac{1}{\nu\mu_1+1}\right)^{\frac{1}{\mu_1}}\left(\frac{1}{s+1}\right)^{\frac{1}{\mu_2}}\left\{\frac{(k_1^{\sigma'} - m^{\sigma'}w_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|^{\mu_2} + m^{\sigma'}\left|\Phi_1'(w_1^{\sigma'})\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right.$$
$$\left. + \frac{(k_1^{\sigma'} - m^{\sigma'}v_1^{\sigma'})}{2}\left[\left|\Phi_1'(k_1^{\sigma'})\right|^{\mu_2} + m^{\sigma'}\left|\Phi_1'(v_1^{\sigma'})\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right\}.$$

Corollary 16. If we choose $m = 1$ in Theorem 6, we deduce

$$\left|\Phi_1(k_1^{\sigma'}) - \frac{(\sigma')^{\nu}\Gamma(\nu+1)}{2}\left[\frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^-}\Phi_1(w_1^{\sigma'})}{(k_1^{\sigma'} - w_1^{\sigma'})^{\nu}} + \frac{{}^{\sigma'}\mathbb{I}^{\nu}_{k_1^+}\Phi_1(v_1^{\sigma'})}{(v_1^{\sigma'} - k_1^{\sigma'})^{\nu}}\right]\right|$$
$$\leq \left(\frac{1}{\nu\mu_1+1}\right)^{\frac{1}{\mu_1}}\left(\frac{1}{s+1}\right)^{\frac{1}{\mu_2}}\left\{\frac{(k_1^{\sigma'} - w_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + \left|\frac{\Phi_1'(w_1^{\sigma'})}{e^{\xi_1 w_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right.$$
$$\left. + \frac{(k_1^{\sigma'} - v_1^{\sigma'})}{2}\left[\left|\frac{\Phi_1'(k_1^{\sigma'})}{e^{\xi_1 k_1^{\sigma'}}}\right|^{\mu_2} + \left|\frac{\Phi_1'(v_1^{\sigma'})}{e^{\xi_1 v_1^{\sigma'}}}\right|^{\mu_2}\right]^{\frac{1}{\mu_2}}\right\}.$$

Example 3. If we choose $s = m = 1 \in (0,1]$, $\xi_1 = 0 \in \mathbb{R}$, $\sigma' = 1$, $\eta = \frac{1}{2} \in [0,1]$, $k_1 = \frac{5}{2}$, and $v_1 = 3$ in Theorem 5, then $\Phi_1(t) = t^4$ is an exponentially (s,m)-convex, as Theorem 5 satisfying the following estimation:

$$g(i) = 10 - \frac{195.31 - 2w_1^5}{50 - 20w_1} \leq 0.71\left\{\frac{5 - 2w_1}{4}\left(1302.08 + 2.66w_1^6\right)^{\frac{1}{2}} - 14.24\right\} = k(i). \tag{11}$$

Example 4. If we choose $s = m = 1 \in (0,1]$, $\xi_1 = 0 \in \mathbb{R}$, $\sigma' = 1$, $\eta = \frac{1}{2} \in [0,1]$, $k_1 = \frac{5}{2}$, and $v_1 = 3$ in Theorem 6, then $\Phi_1(t) = t^4$ is an exponentially (s,m)-convex, as Theorem 6 satisfying the following estimation:

$$g(i) = 10 - \frac{195.31 - 2w_1^5}{50 - 20w_1} \leq 0.4082\left\{\frac{5 - 2w_1}{4}\left(3906.25 + 16w_1^6\right)^{\frac{1}{2}} - 31.2\right\} = l(i). \tag{12}$$

Graphical comparison of Inequalities (11) and (12) is presented using MATLAB R2019a software. It is evident from the figure that the gap between the bounds in Inequality (12) is smaller, which indicates that it provides a more accurate estimate (see Figure 3). Furthermore, Table 1 lists the comparision of our obtained results and the previous studies available in article [28].

Table 1. Comparison of our obtained results with the previous findings

Limiting case of parameters	Operator obtained	Form of inequality	Correspnding known result
General case($\sigma' > 0, \nu > 0$)	Katugampola fractional integral ${}^{\sigma'}\mathbb{I}^{\nu}$	New Hermite-Hadamard type inequality (Main theorem)	Present work
$\sigma' \to 1, m = 1$ and $\xi_1 = 0$	Riemann-Liouville fractional integral $\mathbb{I}^{\nu}$	Taking absolute value in Lemma 2	We get Theorem (5) Khan *et al.* [28]
$\sigma' \to 1, m = 1$ and $\xi_1 = 0$	Riemann-Liouville fractional integral $\mathbb{I}^{\nu}$	By applying Power-mean ineqality in Lemma 2	We get Theorem (6) Khan *et al.* [28]
$\sigma' \to 1, m = 1$ and $\xi_1 = 0$	Riemann-Liouville fractional integral $\mathbb{I}^{\nu}$	By applying Hölder ineqality in Lemma 2	We get Theorem (8) Khan *et al.* [28]

4. CONCLUSION

This study effectively broadens Hermite-Hadamard inequalities by incorporating Katugampola fractional integrals, which represent a noteworthy development in the field of fractional calculus. By developing a new integral identity, we have created a framework for thoroughly investigating functions characterized by extended exponentially (s,m)-convex first-order derivatives. Moreover, we provide an example illustrating our theoretical findings, supported by a graphical representation. Hermite-Hadamard type inequalities are powerful tools for establishing bounds for symmetric expressions. When paired with extended exponentially (s, m)-convex functions, these inequalities become more versatile and yield sharper estimates. The contributions of this work make several possible directions for future research possible, including applying these inequalities to fractional integral operators and investigating their relationships with fractional calculus. Additionally, analyzing analogous inequalities for other classes of convex functions, including strong extended exponentially convex methods, strong (α, m)-convex methods. These novel inequalities presented in this research enhance the theoretical framework of fractional Hermite-Hadamard type inequalities, offering greater accuracy, broader applicability, and a solid basis for future studies on exponentially (s, m)-convex functions.

FUNDING INFORMATION

The first author is financially supported by Research grant for Faculty (IoE Scheme) under Deb. Scheme No. 6031.

AUTHOR CONTRIBUTIONS STATEMENT

This journal uses the Contributor Roles Taxonomy (CRediT) to recognize individual author contributions, reduce authorship disputes, and facilitate collaboration.

Name of Author	C	M	So	Va	Fo	I	R	D	O	E	Vi	Su	P	Fu
Dipak Kr Das	✓	✓	✓	✓	✓	✓		✓	✓	✓				
Shashi Kant Mishra		✓				✓		✓	✓	✓	✓	✓		
Pankaj Kumar	✓		✓	✓		✓			✓		✓		✓	
Abdelouahed Hamdi					✓		✓			✓		✓		✓

C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal Analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject Administration
Fu : **Fu**nding Acquisition

CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

DATA AVAILABILITY

Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

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