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Federerer Measures, Good and Nonplanar Functions of Metric Diophantine Approximation

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ABSTRACT
The goal of this paper is to generalize the main results of [1] and subsequent papers on metric Diophantine approximation with dependent quantities to the set-up of systems of linear forms. In particular, we establish "joint strong extremality" of arbitrary finite collection of smooth nondegenerate submanifolds of $\mathbb{R}^n$. The proof was based on quantitative nondivergence estimates for quasi-polynomial flows on the space of lattices.

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1. INTRODUCTION

The theory of simultaneous Diophantine approximation is concerned with the following question: if Y is an $m \times n$ real matrix (interpreted as a system of m linear forms is n variables), how smallin term of the size of $\mathbf{q} \in \mathbb{Z}^n$,can be the distance from Y$\mathbf{q}$ to$\mathbb{Z}^n$ .This generalization of the classical theory of approximation of real numbers by rational numbers ,where $m = n = 1$.

We start from $m = n = 1$ case. In this case,$Y = y$,it is well known that for any $y \in \mathbb{R}$, thereexist infinitely many integers q_i ' s with integers p_i's satisfying the following :

$$|q_i\, y + p_i| < q_i^{-1} ,$$

or equivalently,

$$\left|y + \frac{p_i}{q_i}\right| < q_i^{-2}$$

Here $|\cdot|$ denotes absolute value. This inequality means that we can approximate any real number y by a sequence of rational numbers and the distance between the real number y and the rational number $-p_i/q_i$ is much smaller than q_i^{-2} $i.e$ $q_i^{-2-\delta}$ .However, it turns out that many real numbers donot have such "better approximation" (so called "very well approximation"). In fact, it is known that for any $\delta > 0$and for Lebesgue (later on we will extend this to other measures) almost every$y \in \mathbb{R}$, the following inequality:

$$|qy + p| < q^{-(1+\delta)}$$

does not have infinitely many integer solutions of p, q.

More generally, in the case that $m = 1$ or, dually $n = 1$, i.e. when $Y = (y_1, \cdots, y_n)$ or $Y = (y_1, \cdots, y_m)^T$(here $(\cdot)^T$ is the transpose of a vector or a matrix) the very well approximation property doesnot hold for almost every Y with respect to Lebesgue measure. Moreover, during recent years, significant progress has been made in showing that not very well approximation properties of vectors⁄forms happen to be generic with respect to certain measures besides Lebesgue measure. This circle of problems dates back to the 1930s, namely, to K. Mahler's work on a classification of transcendental numbers. Let us introduce some definitions and notions order to state these results a clearer way.

Denote by $M_{m,n}$the space of real matrices with m rows and n columns. Dirrichlet's theorem on simultaneous Diophantine approximation states that for any $Y \in M_{m,n}$ (viewed as a system of m linear forms in n variables) and for any $t > 0$ there exist $\mathbf{q} = (q_1, \cdots, q_n) \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} = (p_1, \cdots, p_m) \in \mathbb{Z}^m$ satisfying the following system of inequalities:

$$\|Y\mathbf{q} + \mathbf{p}\|_\infty < e^{-t/m} \quad \text{and} \quad \|\mathbf{q}\|_\infty \le e^{t/n}$$

Here and here after, unless otherwise specified, $\|\cdot\|_\infty$ stands for the maximal norm on $\mathbb{R}^k$ given by $\|\mathbf{x}\|_\infty = \max_{1\le i\le k} |x_i|$. Another way to state this theorem is the following: For any $Y \in M_{m,n}$there are infinitely many $\mathbf{q} \in \mathbb{Z}^n\{0\}$ and $\mathbf{p} = \mathbb{Z}^m$ such that

$$\|Y\mathbf{q} + \mathbf{p}\|_\infty < \|\mathbf{q}\|_\infty^{-n/m} . \tag{1.1.3}$$

Next, we say that Y is very well approximable (to be abbreviated by VWM) if for some positive δ,

$$\exists\infty\, many\ \mathbf{q} \in \mathbb{Z}^n \setminus \{0\} \text{ and } \mathbf{p} \in \mathbb{Z}^m \text{ with } \|Y\mathbf{q} + \mathbf{p}\|_\infty < \|\mathbf{q}\|_\infty^{-n/m-\delta}. \tag{1.1.4}$$

One can show that Lebesgue-a.e Y is not VWA by Borel-Cantelli lemma (we will introduce this lemma in our next section). Also note that by Khintchine's Transference Principle, see e.g. [8, Chapter V], Y is VWM iff the transpose of Y is.

With these definitions and notations, let us go back to one theorem conjectured by Mahler [9] in 1932 and proved three decades later by V. Sprindžuk, see [7, 6], which states that for λ –a.e. $.x \in \mathbb{R}$, the row vector.

$$\mathbf{f}(x) = (x, x^2, \cdots, x^n) \tag{1.1.5}$$

is not VWM.

By extending this problem into a more general setting, that is, for $\mathbf{x} \in \mathbb{R}^d$, by definition:

$$\mathbf{f}(\mathrm{x}) = (\mathrm{f}_1(\mathrm{x}), \cdots, \mathrm{f}_\mathrm{n}(\mathrm{x})),$$

With f_i's continuous maps from $\mathbb{R}^d$ to $\mathbb{R}$, one ask whether or not for almost every $\mathrm{x} \in \mathbb{R}^d$, f(x) is not VWA, with respect to Lebesgue measure or some other measures. In this setting, [4] proved the result for Lebesgue Measure and nondegenerate map f and [1] proved the result for more general assumptions on measures and maps .Before we state result of [4], recall that a smooth map f from $U \subset \mathbb{R}^d$ to $\mathbb{R}^n$ is called nondegenerate at $\mathbf{x} \in$ U if partial derivatives of f at x span $\mathbb{R}^n$ and f is nondegenerate if it is nondegenerate at λ-a.e.x∈U.

Theorem 1.1.1. Let ν be a Federer measure on $\mathbb{R}^d$,U an open subset of $\mathbb{R}^d$, and F:U $\to M_{m,n}$ a continuous map such that $(\mathrm{D}_{\mathrm{F}}, \nu)$ is good and nonplanar .Then for ν-a.e $\mathbf{x}_0 \in$U there exist a ball B ⊂ U centered at $\mathbf{x}_0$ and C , α > 0 such that for any $\mathbf{t} = (\mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{m}, \mathrm{t}_{\mathrm{m}+1}, \ldots, \mathrm{t}_{\mathrm{m}+\mathrm{n}}) \in \boldsymbol{\alpha}^+$ and any $\varepsilon > 0$,

$$\nu(\{\mathrm{x} \in \mathrm{B}: g_{\mathbf{t}}\bar{\tau}(\mathrm{F}(\mathrm{x})) \notin K_\varepsilon\}) < \mathrm{C}\,\varepsilon^\alpha.$$

We will prove this theorem later. Next, let us introduce Borel- Cantelli lemma:

Lemma 1.1.2(Borel- Cantelli).Let ν be a finite measureon B, i.e. ν(B) <∞. If $\{E_i\}_{i=1}^\infty$ is a sequence of sets in B such that.

$$\sum_{i=1}^{\infty} v(E_i) < \infty,$$

Then it follows that:

$$v(\limsup_{i\to\infty} E_i) = 0$$

Recall that

$$\limsup_{i\to\infty} E_i = \bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty} E_i,$$

i.e. each element of the $\limsup_{i\to 1} E_i$ belong to infinitely many E_i's.

2. QUANTITATIVE NON DIVERGENCE

In the section we will state Theorem 2.2.2 that has been proved in [1] and apply it to prove Theorem 1.1.1.In order to state Theorem 2.2.2 we need to introduced some notations and definitions.

Recall that $\{\mathbf{e}_1, \cdots, \mathbf{e}_m, \mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a standard basis of $\mathbb{R}^{m+n}$. Define $M \stackrel{\text{def}}{=} \{1, \cdots, m\}$ and $N \stackrel{\text{def}}{=} \{1, \cdots, n\}$. The following notations will be introduced on the exterior algebra of $\mathbb{R}^{m+n}(\bigwedge(\mathbb{R}^{m+n}))$.

Take

$$I = \{i_1, \cdots, i_r\} \subset M \text{ with } i_1 < \cdots < i_r \text{ and } J = \{j_1, \cdots, j_s\} \subset N \text{ with } j_1 < \ldots < j_s \tag{2.2.1}$$

and denote $\boldsymbol{e_I} \stackrel{\text{def}}{=} \mathbf{e}_{i_1} \wedge, \cdots, \wedge\, \mathbf{e}_{i_r}$ and $\mathbf{v}_J \stackrel{\text{def}}{=} \mathbf{v}_{j_1} \wedge, \cdots, \wedge \mathbf{v}_{j_s}$, with the convention $\mathbf{e}_\emptyset = \mathbf{v}_\emptyset = 1$. Denote by $|I|$ the cardinality of set I, so that $\mathbf{e}_I \wedge \mathbf{v}_J$'s are the basis elements of $\bigwedge^{|I|+|J|}(\mathbb{R}^{m+n})$. We say that a subspace of $\mathbb{R}^{m+n}$ is rational if it is spanned by vectors with rational coordinates or equivalently integer coordinates. Define: $\mathcal{W} \stackrel{\text{def}}{=}$ the set of nonzero rational subspaces of $\mathbb{R}^{m+n}$.

For $g \in$ G (recall that G=$SL_{m+n}(\mathbb{R})$) and W $\in \mathcal{W}$, let $\{\mathbf{w}_1, \cdots, \mathbf{w}_l\}$ be a generating set for $\mathbb{Z}^{m+n} \cap$ W, i. e . $\mathbb{Z}^{m+n} \cap$ W = $span_{\mathbb{Z}}(\boldsymbol{w}_1, \cdots, \boldsymbol{w}_l)$, and define the g action on $\mathbf{w}_1 \wedge \ldots \wedge \mathbf{w}_l$ by

$$g(\mathbf{w}_1 \wedge \ldots \wedge \mathbf{w}_l) \stackrel{\text{def}}{=} g(\boldsymbol{w}_1) \wedge \ldots \wedge g(\mathbf{w}_l)$$

We will write w = $\mathbf{w}_1 \wedge \ldots \wedge \mathbf{w}_l$ and we will say that w corresponds to the nonzero rational subspaces W. Then define $\ell_w(g)$ as the covolume of $gW \cap g\mathbb{Z}^{m+n}$ in gW, i.e.

$$\ell_w(g) \stackrel{\text{def}}{=} \|g(\mathbf{w})\| = \|g(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_l)\|. \tag{2.2.2}$$

The norm $\|\cdot\|$ is an extension of Euclidean norm of $\mathbb{R}^{m+n}$, i .e. if $\mathbf{w}' \in \bigwedge^l(\mathbb{R}^{m+n})$ can be written as

$$w' = \sum_{I\subset M, J\subset N: |I|+|J|=l} a_{I,J}\, \mathbf{e}_I \wedge \mathbf{v}_J,$$

where $a_{I,J}$'s are coefficient and I, J are define in (1.2.1), then

$$\|\mathbf{w}'\| = \sqrt{\sum_{|I|+|J|=l} a_{I,J}^2}.$$

Note that $\ell_w(g)$ independent of the choice of a generating sets. Now, let us state Theorem 2.2.2 as follows: Theorem 2.2.2 (Theorem 4.3 of [1]). Let C''', D, α be positive constants .Suppose U⊂$\mathbb{R}^d$ is open, v is a measure which is D - Federer on U ,h is a continuous map U → G , 0 < ϱ ≤ 1 ,$\mathbf{x}_0 \in$ U ∩ supp v, and B =B ($\mathbf{x}_0$, r) is a ball such that $\tilde{B} \stackrel{\text{def}}{=}$ B($\mathbf{x}_0$,3^{m+n}r) is continuous in U , for each W ∈ $\mathcal{W}$,

(1) The function $\ell_w \circ h$ is (C''', α) -good on $\tilde{B}$ with respect to v, and

(2) $\|\ell_W \circ h\|_{v,B} \geq \varrho$,

Then there exist C'''>0 such that for any 0 <ε ≤ ϱ,

$$v\,(\{\,\mathbf{x} \in \mathrm{B}: \pi\,(h(\mathrm{x})) \notin K_\varepsilon\}) \leq c' \left(\frac{\varepsilon}{\varrho}\right)^{\alpha} v(\mathrm{B}\,).$$

This theorem is known as nondivergence theorem.

In [4, Theorem 5.2], D. Kleinbock and G. Margulis proved this result for Lebesgue measure λ, following the idea of [5, Main Lemma]. Another version which replaces condition (2) by weaker conditions appeared in [3] .The proof of Theorem 2.2.2 which assume v is D-Federer measure is in the paper [1]. Before applying The-orem 2.2.2 to prove this theorem 1.1.1, let us define expanding basis elements : Fix t ∈ $\mathbf{a}^{+}$, and say that basis element of $\bigwedge(\mathbb{R}^{m+n})$,$\mathbf{e}_I \wedge \mathbf{v}_J$ is expanded by g_t (g_t is define as)

$$g_t = \mathrm{diag}\,(e^{t_1}, \cdots, e^{t_m}, e^{-t_{m+1}}, \cdots, e^{-t_{m+n}}),\ \text{where } \mathrm{t} = (t_1, \ldots, t_{m+n}) \in \mathbf{a}^{+}$$

If

$$\|g_{\mathbf{t}}(\mathbf{e}_I \wedge \mathbf{v}_J)\| \geq \|\mathbf{e}_I \wedge \mathbf{v}_J\|.$$

In this case we say that $\boldsymbol{e}_I \wedge \mathbf{v}_J$ is an expanding basis element.Moreover, we say that $g_t$strictly expand $\mathbf{e}_I \wedge \mathbf{v}_J$ if

$$\|g_{\mathbf{t}}(\mathbf{e}_I \wedge \mathbf{v}_J)\| > \|\boldsymbol{e}_I \wedge \mathbf{v}_J\|.$$

On other hand, we say that $g_{\mathbf{t}}$ strictly contracts $\mathbf{e}_I \wedge \mathbf{v}_I$ if it does not expand$\mathbf{e}_I \wedge \mathbf{v}_J$ .Let $t_I = \sum_{i \in I} t_i$ and $t_J = \sum_{j \in J} t_{j+m}$.Clearly, $g_{\mathbf{t}}$ strictly expands $\mathbf{e}_I \wedge \mathbf{v}_J$if $t_I - t_J > 0$ since $\|g_{\mathbf{t}}(\mathbf{e}_I \wedge \mathbf{v}_J)\| = e^{t_I - t_J}\|\mathbf{e}_I \wedge \mathbf{v}_J\|$. In this case, we define the subspace of $\bigwedge(\mathbb{R}^{m+n})$ generated by $\mathbf{e}_I \wedge \mathbf{v}_J$'s with $t_I - t_J > 0$ as strictly expanding subspace ,and denote by E^{+}.Similarly, the subspace generated by $\mathbf{e}_I \wedge \mathbf{v}_J$'s with $t_I - t_J = 0$ is denoted by E^{0} and the subspace generated by $\mathbf{e}_I \wedge \mathbf{v}_J$'s with $t_I - t_J < 0$ is denoted by E^{-}. So we can decompose the space $\bigwedge(\mathbb{R}^{m+n}) = E^{+} \oplus E^{0} \oplus E^{-}$,Where $g_{\mathbf{t}}$ strictly expands the norm of the elements in E^{+} ,does not change the norm of elements in $E^{0}$and contracts the norm of elements in E^{-}. One fact is thatE^{+}, $E^{0}$and E^{-} are dependent on t so that different t give different decomposition of $\bigwedge(\mathbb{R}^{m+n})$. If w ∈ $\bigwedge(\mathbb{R}^{m+n})$ can be written as w = $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in E^{+} \oplus E^{0}$ and $\mathbf{w}_2 \in E^{-}$, then we say $\boldsymbol{w}_1$ is the expanding part of w and $\mathbf{w}_2$ is the contracting part of w.Similarly, expanding part and contracting part are also dependent on t. We will use the definitions and notations above to solve theorem in this paper.

3. PROOF OF MAIN RESULT

Proof of Theorem 1.1.1. Let h(x) = $g_t(\tau\,(\mathrm{F}(\mathrm{x}))$ and for v–a.e. $\mathbf{x}_0$, take

$$\tilde{B} = \mathrm{B}\,(\mathbf{x}_0, 3^{m+n}\mathrm{r}) \subset \mathrm{U}$$

such that (D_F, v) is (C , α)-good on $\tilde{B}$ for some C, α >0 and nonplanar on B = B ($\mathbf{x}_0$, r). To apply Theorem 2.2.2, we need to show that there exist C''', α >0 and some 0 < ϱ ≤ 1 such that for any W ∈ $\mathcal{W}$ where $\mathcal{W}$ is the set of nonzero rational subspaces: $\ell_w \circ$h is (C''', α)-good on $\tilde{B}$ with respect to v (condition (1)) and $\|\ell_w \circ h\|_{v,B} \geq \varrho$ (condition(2)).

First, we want to show that there exist C''' , α > 0 such that for any W ∈ $\mathcal{W}$, $\ell_w \circ$h is (C''' , α)-good on $\tilde{B}$ with respect to v.That is, we want to show that for { $\mathbf{w_1}$,…, $\boldsymbol{w}_l$} a generating set of rational subspace W (we assume it is l dimensional rational subspace) and for w = $\mathbf{w}_1 \wedge \ldots \wedge \mathbf{w}_l$ a homogeneous element of $\wedge^l(\mathbb{R}^{m+n})$ corresponding to W, $\|g_{\mathbf{t}}(\tau(F(\mathbf{x})))(\mathbf{w})\|$ is (C''' , α)-good on $\tilde{B}$ with respect to$v$. The strategy is as follow: first, we want to apply τ(F($\mathbf{x}$)) to w and calculate the result. After calculating τ (F(x)) (w), we will use it to show $\left(\left\|g_{\mathbf{t}}\ \left(\tau\big(F(\mathbf{x})\big)\right)(\mathbf{w})\right\|, v\right)$is ($C'''$, α) -good on $\tilde{B}$.

To calculate τ (F($\mathbf{x}$))($\mathbf{w}$), let us first apply τ(F($\mathbf{x}$)) to basis elements of $\bigwedge(\mathbb{R}^{m+n})$. For reader's convenience, we start by applying τ (F(x)) to basis elements of $\bigwedge(\mathbb{R}^{m+n})$ of dimension 1 and dimension 2:

For dimension 1, apply τ (F (x)) to basis elements $\mathbf{e}_i$ and $\mathbf{v}_j$ for $i = 1, \cdots, m$ and j=1, $\cdots, n$:

$$\tau\,(\mathrm{F}(\mathbf{x}))\,(\mathbf{e}_i) = \mathbf{e}_i$$

for $i = 1, \cdots, m$.

τ (F(x))($\mathbf{v}_j$) = $\mathbf{v}_j$ = f_{1_j}(x)$\mathbf{e}_1$ + . . . + f_{m_j}(x)$\mathbf{e}_m$

for j = 1, ⋯, n.

So τ (F (x)) fixes $\mathbf{e}_i$ and when applying τ (F(x)) to $\mathbf{v}_j$, one gets a linear combination of elements of dimension 1 and the coefficients in front of each element are either 1 or some components f_{ij}(x)' s in F(x).

For dimension 2, there are three different types of basis elements, they are $\mathbf{e}_i\wedge \mathbf{e}_j$,$\mathbf{v}_i\wedge \mathbf{v}_j$ and $\mathbf{e}_i\wedge \mathbf{v}_j$. We want to apply τ (F(x)) to all types of basis elements:

$$\tau (F(\mathbf{x}))(\mathbf{e}_i\wedge\mathbf{e}_j) = \mathbf{e}_i\wedge \mathbf{e}_j$$

for any $1 \le i, j \le m$.

$$\tau(F(\mathbf{x}))(\mathbf{v}_i\wedge\mathbf{v}_j) = \mathbf{v}_i\wedge\mathbf{v}_j + \sum_{l=1}^{m} f_{lj}(\mathbf{x})\mathbf{e}_l\wedge\mathbf{v}_i - \sum_{l=1}^{m} f_{li}(\mathbf{x})\boldsymbol{e}_l\wedge\mathbf{v}_j + \sum_{1\le k<l\le m} \begin{vmatrix} f_{ki}(\mathbf{x}) & f_{li}(\mathbf{x}) \\ f_{kj}(\mathbf{x}) & f_{lj}(\mathbf{x}) \end{vmatrix} \mathbf{e}_k\wedge\mathbf{e}_l$$

for any $1 \le i,j \le n$.

$$\tau(F(\mathbf{x}))(\mathbf{e}_i\wedge\mathbf{v}_j) = \mathbf{e}_i\wedge\mathbf{v}_j + \sum_{\substack{1\le l\le m \\ l\ne i}} f_{lj}(\mathbf{x})\mathbf{e}_l\wedge\boldsymbol{e}_j$$

for any $1 \le i \le m, 1 \le j \le n$.

Similarly as for dimension 1, we can conclude that the result of applying τ(F(x) to any type of basis element is a linear combination of basis elements of dimension 2 with coefficient are 1, $f_{ij}(\mathbf{x})$s or the determinants of 2 by 2 submatrices of F(x). These observations lead us to the general l results: by applying τ(F(x)) to any basis elements of dimension $\wedge^l(\mathbb{R}^{m+n})$, the result is a linear combination of basis elements of dimension $\wedge^l(\mathbb{R}^{m+n})$ with the coefficients are 1or determinants of submatrices of $F(\mathbf{x})$. To prove this, let |I | be the cardinality of set I and $\mathbf{e}_I\wedge\mathbf{v}_J$ be a basis element of dimension $l = |I| + |J|$. Applying $\tau(F(\mathbf{x}))$ to $\mathbf{e}_I\wedge\mathbf{v}_J$, one has:

$$\tau(F(\mathbf{x}))(\mathbf{e}_I\wedge\mathbf{v}_J) = \sum_{S\subset J} \sum_{\substack{K\subset M\setminus I, \\ |K|=|S|, L=J\setminus S}} (-1)^{m(I,K)} f_{K,S}(\mathbf{x})\, \mathbf{e}_{I\cup K}\wedge\mathbf{v}_L \tag{2.2.3}$$

Where $f_{K,J}(\mathbf{x})$'s defined as:

$$f_{I,J} \stackrel{\text{def}}{=} \begin{vmatrix} f_{i_1,j_1} & \cdots & f_{i_1,j_r} \\ f_{i_r,j_1} & \cdots & f_{i_r,j_r} \end{vmatrix},$$

And m (·) determines the sign of the coefficients.Clearly, $\tau(F(\mathrm{x}))$ ($\mathbf{e}_I\wedge\mathbf{v}_J$) is a linear combination of basis elements of $\wedge^l(\mathbb{R}^{m+n})$ and the coefficients of these basis elements are determinants of square submatrices of F($\mathbf{x}$).

Now, let us apply $\tau(F(\mathbf{x}))(\mathbf{e}_I\wedge\mathbf{v}_J)$ is a linear combination of basis elements of $\wedge^l(\mathbb{R}^{m+n})$ and the coefficient of these basis elements are determinants of square submatrices of F(x).Now, let us apply $\tau(F(\mathbf{x}))$ to w where w is a homogenous element of $\wedge^l(\mathbb{R}^{m+n})$ corresponding to l dimension rational subspace W. Since $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l\}$ is a generating set of nonzero rational subspace W, we can write w = $\mathbf{w}_1\wedge \ldots \wedge \mathbf{w}_l$ = $\sum_{I,J} a_{I,J}\, \mathbf{e}_I\wedge\mathbf{v}_J \in \wedge^l(\mathbb{R}^{m+n})$, where the summation is over all I, J satisfying (2.2.1) with l = |I | + | J | is fixed and $a_{I,J}$'s are integers coefficients. By equation (2.2.3) since w is a linear combination of $\mathbf{e}_I\wedge\mathbf{v}_J$'s τ(F($\mathbf{x}$)) ($\mathbf{w}$) is also a linear combination of basis elements $\wedge^l$ ($\mathbb{R}^{m+n}$) with the coefficients of each $\mathbf{e}_I\wedge\mathbf{v}_J$ in τ(F($\mathbf{x}$))($\mathbf{w}$) a combination of $f_{K,J}$(x)'s. Let $\omega'_{I,J}(\mathbf{x})$ be the coefficient of $\mathbf{e}_I\wedge\mathbf{v}_J$ in τ (F($\mathbf{x}$))($\mathbf{w}$), then we can write:

$$\tau(F(\mathbf{x}))(\mathbf{w}) = \sum_{I,J} \omega'_{I,J}(\mathbf{x})\, \mathbf{e}_I\wedge\mathbf{v}_J\ .$$

Now, let us apply $g_{\mathbf{t}}$, for **t** in $\mathbf{a}^{+}$. Take $t_I = \sum_{i\in I} t_i$ and $t_J = \sum_{j\in J} t_{m+j}$. We have:

$$g_{\mathbf{t}}(\tau(\mathrm{F}(\mathbf{x})))\mathbf{w} = \sum_{I,J} e^{t_I - t_J}\omega'_{I,J}(\mathbf{x})\mathbf{e}_I \wedge \mathbf{v}_J.$$

Let $\omega_{I,J}(\mathrm{x}) = e^{t_I - t_J}\omega'_{I,J}(\mathrm{x})$. By the assumption that (D_F, v) is (C,α)-good on $\tilde{B}$, those $\omega_{I,J}(\mathrm{x})$'s which are linear combination of $f_{I,J}$'s are also (C,α)-good on $\tilde{B}$ with respect to v. Furthermore, this implies that $\mathrm{Q}(\mathrm{x}) \stackrel{\text{def}}{=} \|g_{\mathbf{t}}(\tau(F(\mathbf{x})))\mathbf{w}\|$ is (C''', α)-good on $\tilde{B}$ with respect to v for any homogenous w and some C''',α >0. To show this, let $\mathbf{e}_{I^*}\wedge\mathbf{v}_{J^*} \in \wedge^{|I|+|J|}(\mathbb{R}^{m+n})$ be such that the coefficient $\omega_{I^*,J^*}(\mathrm{x})$ satisfies the following:

$$\|\omega_{I^*,J^*}(\mathbf{x})\|_{v,\tilde{B}} = \max_{|I|+|J|=l}\{\|\omega_{I,J}(\mathbf{x})\|_{v,\tilde{B}}\},$$

where the maximum is taken among all the norms of coefficients of $\tau(\mathrm{F}(\mathbf{x})(\mathbf{w})$. Following the definition of the norm ||·|| on the exterior algebra $\bigwedge(\mathbb{R}^{m+n})$,

$$\|Q(\mathbf{x})\|_{v,\tilde{B}} = \| \sqrt{\sum_{|I|+|J|=l} \omega_{I,J}^2(\mathbf{x})} \|_{v,\tilde{B}} \le \sqrt{\sum_{|I|+|J|=l} \|\omega_{I,J}^2(\mathbf{x})\|_{v,\tilde{B}}}.$$

Then it is clear that there exist some constant $C'''\geq 0$ (C'''depends on m , n but not on **w**) such that the following is satisfied:

$$\|\omega_{I^*,J^*}(\mathbf{x})\|_{v,\tilde{B}} \geq C'''\|Q(\mathbf{x})\|_{v,\tilde{B}} .$$

It follow that there exist some $C'''> 0$ such that:

$$v(\{\mathbf{x} \in \tilde{B} : |\, \mathrm{Q}(\mathbf{x})| < \varepsilon\}) \le v(\{\mathbf{x} \in \tilde{B}: |\, \omega_{I^*,J^*}(\mathbf{x})| < \varepsilon\})$$

$$\le \mathrm{C}\left(\frac{\varepsilon}{\|\omega_{I^*,J^*}(\mathbf{x})\|_{v,\tilde{B}}}\right)^{\alpha} v(\tilde{B}) = C'''\left(\frac{\varepsilon}{\|Q(\mathbf{x})\|_{v,\tilde{B}}}\right)^{\alpha} v(\tilde{B}).$$

The second inequality following from (C, α)-good property of $\omega_{I^*,J^*}(\mathbf{x})$ on $\tilde{B}$ with respect to v. this proves that (Q ($\mathbf{x}$),v) is (C''', α)-good on $\tilde{B}$ for any **w** and consequently, $\ell_w \circ h$ (C''', α)-good on $\tilde{B}$ with respect to v for any nonzero rational subspace W and h(**x**) = $g_{\boldsymbol{t}}$(τ (**x**)).

Now, let us show that the second condition is satisfied; that is: there exist some ϱ with $0 < \varrho \le 1$ such that for any nonzero rational subspace W of $\mathbb{R}^{m+n}$ and any **t**,

$$\|\ell_{w}\circ h\|_{v,\tilde{B}} \ge \varrho,$$

when $h(\mathbf{x}) = g_{\boldsymbol{t}}(\tau(\mathrm{F}(\mathbf{x})))$.

Recall from the paragraph before the proof of Theorem 1.1.1 that, an expanding basis element is a basis element $\mathbf{e}_I\wedge\mathbf{v}_J$ such that

$$\|g_{\mathbf{t}}(\mathbf{e}_I \wedge \mathbf{v}_J)\| \geq \|\mathbf{e}_I \wedge \mathbf{v}_J\|,$$

or equivalently,

$$\mathbf{e}_I \wedge \mathbf{v}_J \in \boldsymbol{E}^{+} \oplus \boldsymbol{E}^{0}.$$

Our strategy to show condition (2) of Theorem 2.2.2 is as follow. First, apply τ(F ($\mathbf{x}$)) to $\mathbf{w} = \sum_{I,J} a_{I,J}\mathbf{e}_I \wedge \mathbf{v}_J$ and calculate the result; next take the expanding Part of τ(F($\mathbf{x}$)) ($\mathbf{w}$) and show $\|\ell_w \circ h\|_{v,B} \geq \varrho$ for any nonzero rational subspace W when $h(\mathbf{x}) = g_{\mathbf{t}}(\tau(\mathrm{F}(\mathbf{x})))$. Let us consider two cases:

Case1.When $l \leq m$ denote by E the space generated by $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ and by V the space generated by $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, and consider a projection P : $\bigwedge^l (\mathbb{R}^{m+n}) \longrightarrow \bigwedge^l (E)$.

From the definition of P, the image of P consists of linear combination of expanding basis elements (i. e. elements in $E^+ \oplus E^0$) since $\bigwedge^l$(E) = span $(\mathbf{e}_L)_{L\subset M}$, and for any L ⊂ M, $\|g_t\mathbf{e}_L\| = e^{t_L}\|\mathbf{e}_L\| \geq \|\mathbf{e}_L\|$ where $t_L = \sum_{i\in L} t_i$.Using equation (2.2.3), it is easy to see that

$$\mathrm{P}(\tau(\mathrm{F}(\mathrm{x}))(\mathbf{e}_I \wedge \mathbf{v}_J)) = \mathbf{e}_I \wedge \sum_{K\subset M\setminus I, |K|=|J|} (-1)^{m(I,J)} f_{K,J}(\mathrm{x})\, \mathbf{e}_K, \qquad (2.2.4)$$

Where $K = \{k_1, \cdots, k_s\} \subset M$.

Note that $|I|$ can take values between $\max(0, l-n)$ and l ; equivalently, $l - |I|$ ranges between 0 and $l - \max(0, l-n) = \min(l, n)$.Then we have:

$$\mathrm{P}(\tau(\mathrm{F}(\mathbf{x}))(\mathbf{w})) = \sum_{\substack{I\subset M \\ \max(0,l-n)\leq |I| \leq l}} \mathbf{e}_I \wedge \sum_{\substack{J\subset N \\ |J|=l-|I|}} a_{I,J} \sum_{\substack{K\subset M\setminus|I| \\ |K|=|J|}} (-1)^{m(I,J)} f_{K,J}(\mathbf{x})\, \boldsymbol{e}_K.$$

Rearranging terms and substituting $L = I \cup K$, we get

$$\mathrm{P}(\tau(\mathrm{F}(\mathbf{x}))(\mathbf{w})) = \sum_{\substack{L\subset M \\ |L|=l}} \left(\sum_{\substack{I\subset L \\ \max(0,l-n)\leq|I|\leq l}} \sum_{\substack{J\subset N \\ |J|-l-|I|}} (-1)^{m(I,J)} a_{I,J} f_{L\setminus I,J}(\mathbf{x}) \right) \mathbf{e}_L,$$

Or equivalently,

$$\mathrm{P}(\tau(\mathrm{F}(\mathbf{x}))(\mathbf{w})) = \sum_{\substack{L\subset M \\ |L|=l}} \left(\sum_{\substack{K\subset M \\ 0\leq|K|\leq\min(l,n)\leq|I|\leq l}} \sum_{\substack{J\subset N \\ |J|=|K|}} (-1)^{m(I,J)} a_{L\setminus K} f_{K,J}(\mathbf{x}) \right) \mathbf{e}_L .$$

From the definition of $\ell_w(h(\mathbf{x}))$ for any W and for $h(\mathbf{x}) = g_{\mathbf{t}}(\tau(\mathrm{F}(\mathbf{x})))$, we know that

$$\|\ell_w(h(\mathbf{x}))\|_{v,B} = \left\| \|g_{\mathbf{t}}(\tau(F(\mathbf{x}))(\mathbf{w}))\| \right\|_{v,B},$$

Where in the right hand side of equation, the inner $\|\cdot\|$ is exterior algebra of $\mathbb{R}^{m+n}$ and $\|\cdot\|_{v,B}$ the norm define in following equation,

$$\|f\|_{v,B} = \sup\{c : v(\{\mathbf{x} \in \mathrm{B} : |f(\mathbf{x})| > c\}) > 0\}.$$

From the definition of $\|\cdot\|$ norm and the fact that $\mathbf{e}_L$ is an expanding basis element, $\left\|\|g_{\mathbf{t}}\,\tau(Y)(\mathbf{w})\|\right\|_{v,B}$ is greater than or equal to the $\|\cdot\|_{v,B}$ norm of the coefficients of any $\mathbf{e}_L$, i.e.

$$\left\| \|g_{\mathbf{t}}(\tau(F(\mathbf{x}))(\mathbf{w}))\| \right\|_{v,B}$$

$$\geq \max_{\substack{L\subset M \\ |L|=l}} \| \| g_{\mathbf{t}}\, ((\sum_{\substack{K\subset L \\ 0\leq|K|\leq\min(l,n)\leq|I|\leq l}} \sum_{\substack{J\subset N \\ |J|=|K|}} (-1)^{m(I,J)} a_{L\setminus K,J} f_{K,J}(\mathbf{x}) \mathbf{e}_{\mathrm{L}} \| \|_{v,B}$$

$$\geq \max_{\substack{L\subset M\\|L|=l}} \| \sum_{\substack{K\subset L\\0\leq|K|\leq\min(l,n)\leq|I|\leq l}} \sum_{\substack{J\subset N\\|J|=|K|}} (-1)^{m(I,J)} a_{L\setminus K,J} f_{K,J}(\mathbf{x}) \|_{v,B}.$$

It remains to show that there exist $0 \leq \varrho \leq 1$ such that for any $\mathbf{w} = \sum_{I,J} a_{I,J}\ \mathbf{e}_I \wedge \mathbf{v}_J$ corresponding to a nonzero rational subspace W,

$$\max_{\substack{L\subset M\\|L|=l}} \| \sum_{\substack{K\subset L\\0\leq|K|\leq\min(l,n)\leq|I|\leq l}} \sum_{\substack{J\subset N\\|J|=|K|}} (-1)^{m(I,J)} a_{L\setminus K,J} f_{K,J}(\mathbf{x}) \|_{v,B} \geq \varrho. \tag{2.2.5}$$

From the assumption that (D_F, v) is nonplanar, $f_{K,J}(\mathbf{x})$'s linearly independent on B. And this implies that there exist $0 < \varrho \leq 1$ such that for any $b_{I,J}$'s where $I \subset M, J \subset N$, $|I| = |J| \leq \min(m,n)$ with $\max_{I,J}\{|b_{I,J}|\} \geq 1$, we have:

$$\| \sum_{I,J} b_{I,J}\ f_{I,J}(\mathbf{x}) \|_{v,B} \geq \varrho.$$

From this fact unless $a_{I,J} = 0$ for all $I \subset M$, $J \subset N$ and $|I| = |J| \leq \min(m,n)$ equation (2.2.5) is satisfied. If $a_{I,J} = 0$ for all $I \subset M$, $J \subset N$, and $|I| = |J| \leq \min(m,n)$, then $\mathbf{w}=0$ and the corresponding subspace W is zero .This contradicts to the assumption that W is nonzero. This shows that for the case 1, condition (2) in Theorem2.2.1 is satisfied.

Case2. If $l \geq m$, we need to consider the projection P' from $\Lambda^l(\mathbb{R}^{m+n})$ onto $\mathbf{e}_M \wedge \Lambda^{l-m}(V)$. Similarly, the image of the projection consists of linear combinations of expanding basis elements since for any $L \subset N$,

$$\|g_t(\mathbf{e}_M \wedge \mathbf{v}_L)\| \geq e^{t_M - t_L}\|\mathbf{e}_M \wedge \mathbf{v}_L\| = e^{t-t_L}\|\mathbf{e}_M \wedge \mathbf{v}_L\| \geq \|\mathbf{e}_M \wedge \mathbf{v}_L\|.$$

Then from equation (2.2.3):

$$P'(\tau(F(\mathbf{x}))(\mathbf{e}_I \wedge \mathbf{v}_J)) = \mathbf{e}_I \wedge \left(\sum_{K\subset J, |K|=m-|I|} (-1)^{m(I,J)} f_{M\setminus I,K}(\mathbf{x}) \mathbf{e}_{M\setminus I} \wedge \mathbf{v}_{J\setminus K} \right)$$

$$= \mathbf{e}_M \wedge \left(\sum_{K\subset J, |K|=m-|I|} (-1)^{m(I,J)} f_{M\setminus I,K}(\mathbf{x}) \mathbf{v}_{J\setminus K} \right). \tag{2.2.6}$$

Note that now we must have $\max(0, l-n) \leq |I| \leq m$, or, equivalently, $0 \leq |M \setminus I| \leq m - \max(0, l-n) = \min(m, m+n-l)$ .Therefore:

$$P'(\tau(F(\mathbf{x}))(\mathbf{w})) = \mathbf{e}_M \wedge \sum_{\substack{I\subset M\\ \max(0,l-n)\leq|I|\leq m}} \sum_{\substack{J\subset N\\|J|=l-|I|}} a_{I,J} \sum_{\substack{K\subset J\\|K|=m-|I|}} (-1)^{m(I,J)} f_{M\setminus I,K}(\mathbf{x}) \mathbf{v}_{J\setminus K}.$$

Rearranging terms and substituting $L = J\setminus K$, we get

$$P'(\tau(F(\mathbf{x}))(\mathbf{w}) = \sum_{\substack{L\subset M\\|L|=l-m}} \left(\sum_{\substack{I\subset M\\ \max(0,l-n)\leq|I|\leq m}} \sum_{\substack{J\supset L\\|J|=l-|I|}} (-1)^{m(I,J)} a_{I,J} F_{M\setminus I, J\setminus L}(\mathbf{x}) \right) \mathbf{e}_M \wedge \mathbf{v}_L$$

$$= \sum_{\substack{L\subset N\\|L|=l-m}} \left(\sum_{\substack{I\subset M\\0\le|M\setminus I|\le\min(m,m+n-l)}} \sum_{\substack{K\subset N\setminus L\\|K|=m-|I|}} (-1)^{m(I,J)} a_{I,K\cup L} f_{M\setminus I,K}(\mathbf{x}) \right) \mathbf{e}_M \wedge \mathbf{v}_L .$$

Similarly to the case 1, we want to show that there exist $0 \le \varrho \le 1$ such that for any $\mathbf{w} = \sum_{I,J} a_{I,J} \mathbf{e}_I \wedge \mathbf{v}_J$ corresponding to a nonzero rational subspace W :

$$\max_{\substack{L\subset M\\|L|=l-m}} \| \sum_{\substack{I\subset M\\0\le|M\setminus I|\le\min(m,m+n-l)}} \sum_{\substack{K\subset N\setminus L\\|K|=m-|I|}} (-1)^{m(I,J)} a_{I,K\cup L} f_{M\setminus I,K}(\mathbf{x}) \|_{\nu,B} \ \ge\ \varrho. \qquad (2.2.7)$$

Using the same argument as in case 1, since $a_{I,J} \neq 0$ some I, J and $(D_F\ ,\nu\ )$ is nonplanar on B, the inequality (2.2.7) is satisfied. This shows that in case 2, condition (2) of Theorem 2.2.1 is satisfied. We have shown that both condition (1) and condition (2) of Theorem 2.2.1 are satisfied in Theorem 1.1.1, so we can prove Theorem 1.1.1 by applying Theorem 2.2.1 with $C = C' \left(\frac{1}{\varrho}\right)^{\alpha} \nu\ (B)$.

4. CONCLUSION

In this paper, we studied linear combination, nonplanar condition (C, α)-good function. We apply $\tau(F(\mathbf{x}))$ to $\mathbf{w}$ and calculate the result and after calculating $\tau(F(\mathbf{x}))(\mathbf{w})$apply it to calculating $\left(\left\|g_t\ (\tau(F(\mathbf{x})))(\mathbf{w})\right\|, \nu\right)$ is (C''', α)-good on $\tilde{B}$. We conclude that the result of applying $\tau(F(\mathbf{x}))$ to any type of basis element is a linear combination of basis elements of dimension 2 with the coefficients are 1,f_{ij}(x)s or the determinants of 2 by 2 submatrices of F(x).We have shown that both conditions are satisfied.

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